Three Expansion Regimes for Interest Rate Term Structure Models

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Abstract

We develop expansions for general term structure models in three limits: small volatility, small time to maturity, and large time to maturity. In the small volatility limit, the short-rate evolves deterministically to a stable equilibrium point, which is a zero of the risk-adjusted drift function. In the small time limit, the discount bond price has an asymptotic power series expansion. In the large time limit, the yield equals the smallest eigenvalue of a differential operator, which we calculate using the calculus of variations and other methods. All three expansions apply to multi-factor models as well.

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Introduction

Modern theories of the term structure of interest rates lead to generalized diffusion equations. Coupled with appropriate boundary conditions, the solutions to these equations determine the yield curve for default-free securities at any moment. There are two competing approaches: the arbitrage approach as exemplified by the work of Vasicek (1977) or Richard (1978) or the general equilibrium approach of Merton (1975) or Cox, Ingersoll, and Ross (CIR, 1985). Both approaches in continuous time require that one develop an adequate generalized Brownian motion process for a small number of explanatory factors. Both approaches lead to a generalized diffusion equation of the same structure for the price of a discount bond. Given a model, one can then, in principle, solve that equation, compare the results to reality, and hopefully achieve some insight on what might be improved in the "inputs".

There are many competing models for the precise specification of the drift and volatility of the diffusion process. However, only a few specifications have exact solutions. In this paper we examine three limits in which one can expand *unknown* solutions for the price of discount bonds as systematic expansions in small parameters. The limits are: (i) small factor volatility σ, (ii) small time to maturity τ, and (iii) large time to maturity τ. The expansion parameters for these limits are σ^2, τ, and $\exp(-\tau)$, respectively. Depending upon the type of expansion, the particular details of the model, and the parameter values, these expansions may be exact and convergent, asymptotic and convergent, or asymptotic and not convergent. [See Erdelyi (1956) for discussion of asymptotic expansions]. All these cases occur. Why such expansions are useful and complementary to other numerical approaches is explained below.

For simplicity, we illustrate our methods with single factor models. Even though we use single factor models for ease of exposition, all three expansions have the same validity for multi-factor models. Single versus multi-factor specifications are examined empirically in Stambaugh (1988). Examples of specific two-factor general equilibrium models are presented in CIR (1985), in Longstaff and Schwartz (1992), and further references may be found in Duffie (1992).

In addition, because we are primarily concerned with methods of solution for the diffusion equation, we adopt the point of view and the language of the arbitrage approach. Even though the general equilibrium approach has certain theoretical advantages, the arbitrage approach is more direct for our purposes. In the arbitrage approach, a one-factor theory requires a model in which the short-term rate r (raw interest rate process) follows $dr = m(r)dt + \sigma\sqrt{a(r)}dz$, where $m(r)$ is the drift, dt is calendar time, σ is a volatility scale parameter, $a(r)$ is a volatility function, and dz is a Brownian motion. (Loosely speaking, we shall often just refer to σ^2 as the volatility). The raw interest rate process is the

process that one measures approximately by recording the yields of short-maturity T-bills at discrete calendar time intervals. Because any real bond price will be affected by risk attitudes, this last statement requires some justification, which we provide in Section 2.3. It will often be convenient to write similar expressions for the risk-adjusted interest rate process, which follows $dr = b(r)dt + \sigma\sqrt{a(r)}dz$, where $b(r)$ is the risk-adjusted drift. The two drifts are related by $b(r) = m(r) + \lambda(r)\sigma\sqrt{a(r)}$, where $\lambda(r)$ is the market price of risk for the short-term rate. The market price of risk is that universal ratio at a point in time, for every default-free bond, of the instantaneous excess expected return divided by the instantaneous standard deviation of return.

In single factor models, the price of the default-free discount bond is given by a function $P(r, \tau)$. The first two of our expansions for $P(r, \tau)$ are power series in σ^2 and τ, respectively. The third expansion for $P(r, \tau)$ is an *eigenfunction expansion*, with successive corrections of the form $\exp[-(\lambda_i - \lambda_0)\tau]$, where the λ_i are the eigenvalues of a particular differential operator. In this paper, we work with just the leading term of the eigenfunction expansion.

Consider the limit of a vanishing volatility scale parameter. In this limit, the yield curve is determined by the deterministic time evolution of the short-term rate r. We assume this evolution is controlled by a *stable equilibrium point* r^* of the equation $dx/d\tau = b(x)$, $x(0) = r$. This is a first approximation. This deterministic differential equation is much easier to solve in closed form than the stochastic differential equations of the general theory. We show how to develop corrections to this first approximation in powers of σ^2. It turns out that this volatility series is quite efficient. That is, just a couple of terms provide accurate yield curves out to maturities of several years if the volatility is not too large.

In the limit of small time to maturity τ, the yield curve has an asymptotic power series expansion in τ. With numerical examples, we show that this series requires more terms than the volatility series for a given accuracy. However, the individual terms of the time to maturity expansion are really trivial to compute at low orders. Additionally, this expansion demonstrates that $b(r)$ and $a(r)$ are determined over a range (r_1, r_2) by measurements of the initial slope and initial curvature of the yield curve during the time period in which the short-term rate varies within the indicated range. This partially solves the *inversion* problem for the one factor stationary model, which is the problem to deduce the risk-adjusted short-rate process from the yield curve. In addition, this particular limit clarifies exactly what is being measured when one records the time series of yields for short-maturity T-bills. Is one seeing, for example, the raw interest rate process or the risk-adjusted process?

In the limit of large time to maturity, the yield curve tends to a limit we call R_∞. We show R_∞ to be equal to the smallest eigenvalue λ_0 of the problem $\mathcal{A}f = \lambda f$, where $\mathcal{A}$ is a differential operator acting on functions $f(r)$ in a certain domain. Once this is

established, we illustrate two techniques for determining λ_0. In the first technique, we combine the regimes of small volatility and large time to maturity in order to construct a power series for λ_0 in terms of σ^2. In the second technique, we show how the calculus of variations can be used to construct very accurate estimates for λ_0. These estimates come from an inequality which provides the best possible upper bound to the exact value.

Our expansion methods are complementary to numerical approaches and have the following benefits. (i) Because they are numerically precise in certain limits, they can be used as a check on grid-based numerical approaches. For example, the worst regime for any numerical solution for a term structure model is the limit of infinite time to maturity. This is because numerical solutions proceed sequentially from the boundary condition at zero time to maturity to larger times. In contrast, the case of large time to maturity is the best regime for an eigenvalue expansion because only the smallest eigenvalue contributes in this limit. (See Section 3). (ii) Expansions can be used to produce a quick "first approximation" that one may want to improve later with detailed numerical work; this is a way to use the volatility expansion (See Section 1). (iii) Expansions can give insights into fundamental issues that numerical work cannot. For example, what is the importance and the relationship between the zeroes of the drift term of the raw interest rate process and the risk-adjusted interest rate process? The small time to maturity expansion answers this question. (See Section 2). (iv) Each of the expansions explicitly show the functional dependence of the yield curve on the parameters of the drift and volatility functions. These dependencies cannot be seen in grid-based numerical solutions except by repeated solution of the problem, and one would be unlikely to discover interesting ones such as eq.(1) below (See Section 3). (v) Knowing the exact behavior in certain limits is useful for an error analysis of any numerical method. For example, it is the *difference* between the first two eigenvalues of the eigenfunction expansion that will control convergence of any numerical solution for large times to maturity. (vi) Finally, we believe the first couple terms of the small volatility expansion has pedagogic value in explaining the continuous time term structure theory, in the same sense that the binomial model helps explain the Black-Scholes option theory.

When we want to be more specific than the generalized form $dr = b(r)dt + \sigma\sqrt{a(r)}dz$, we illustrate our techniques on the two classes of models, namely $dr = (A - Br)dt + \sigma r^{\theta} dz$, and $dr = (Ar - Br^2)dt + \sigma r^{\theta} dz$, where $A, B, \sigma,$ and θ are four parameters. The drift direction and rate as well as the risk premium adjustment are determined by A and B. (Note that the scalar drift parameter A is to be distinguished from the operator $\mathcal{A}$). The volatility scale and power behavior are determined by σ and θ. Many of the special cases which have been proposed in the literature are among these two classes; in these special cases, typically θ is integral or half-integral although this is not a requirement of our work here.

Expansion regimes can be combined to arrive at interesting results. For example, consider the combined regime of large time to maturity and small volatility. As we indicated, as the time to maturity increases, the yield to maturity tends to a limit

$R_\infty = \lambda_0$. This limit does not depend on the short-term rate or calendar time. As the volatility parameter tends to zero, we show in Section 3:

$$R_\infty = r^* - \sigma^2 \frac{a(r^*)}{4[b'(r^*)]^2}[2 - b''(r^*)] + O(\sigma^4), \tag{1}$$

where r^* is the assumed stable equilibrium point of the differential equation $dr/d\tau = b(r)$, the primes denote differentiation, and the asymptotic order symbol O is used throughout. (If ε denotes any of our three expansion parameters, then $f(\varepsilon) = O(g(\varepsilon))$ means $\lim f(\varepsilon)/g(\varepsilon)$ is bounded as $\varepsilon \to 0$). To a first approximation, this compact expression (1) shows how the local behavior of the drift and volatility functions in the vicinity of r^* control long-term yields for completely general processes that satisfy our assumptions.

Before proceeding with the mathematical development, we comment briefly on the connection between this work here and certain empirical tests. In reality, the term structure theory is still at an early stage in the sense that not much is really agreed upon about the functions $m(r)$, $\lambda(r)$ and $a(r)$. A contribution along these lines is the recent work by Chan, Karolyi, Longstaff, and Sanders (1992), who estimate parameters for the 4-parameter stationary process $dr = (A - Br)dt + \sigma r^\theta dz$. Using one-month T-bill rates over the period mid-1964 through 1989, they estimate $\theta = 1.50 \pm .25$, (the one standard deviation error is implied by their t-statistics) and $A/B = .068$. First of all, the estimate for θ suggests that more attention should be directed toward models with $a(r) = r^3$, which we provide below in some detail. Secondly, as we noted above, this type of time series study is a measurement of $m(r)$; hence the estimate for A/B is a measurement of the zero of $m(r)$, call it r^{**}. In contrast, we have defined the zero of $b(r)$ to be r^*. In Section 2, we discuss for arbitrary drift specifications the relationship between r^*, r^{**}, and the initial *slope* of the yield curve. The Chan et al study tells us something about $m(r)$ and $a(r)$ but not $\lambda(r)$, nor whether or not the stationary assumption for any of the three functions is an adequate one.[1]

By *yield*, we mean the continuously compounded yield of the discount bond. Price and yield are related by $R(r, \tau) = (-1/\tau)\ln P(r, \tau)$. $P(r, \tau)$ is determined by the arbitrage (or equilibrium) theory to satisfy the partial differential equation

$$\frac{1}{2}\sigma^2 a(r)\frac{\partial^2 P}{\partial r^2} + b(r)\frac{\partial P}{\partial r} - rP = \frac{\partial P}{\partial \tau}. \tag{2}$$

For our work here, we shall take the range of possible values for the short-term rate to be the *natural* one: $0 < r < \infty$, but this is not a requirement of the theory. (For example, a smaller range could be imposed). Since the pure discount bond matures with a value of par, we have the initial condition $P(r, 0) = 1$. Note that the theory is *incomplete* because it does not discuss the nature, if any, of the boundary conditions at the end-points of the

range of acceptable values of the short-term rate. In part, this incompleteness is a product of the indefiniteness of the specifications for the functions $a(r)$ and $b(r)$. That is, depending on the specifications for those functions, there may or may not be the requirement to specify boundary values for $P(r, \tau)$ in order to arrive at a uniquely determined solution to eq. (2). However, here we assume that no boundary conditions are necessary at either $r = 0$ or $r = \infty$; we discuss this point further in Section 3. We are making the following assumptions about the behavior of $P(r, \tau)$, $b(r)$, and $a(r)$. For $P(r, \tau)$, we assume all the derivatives in eq. (2) exist. In addition, we employ these assumptions:

(i) the solution $x(\tau)$ to $dx = b(x)d\tau + \sigma\sqrt{a(x)}dz,\ x(0) = r$
is non-negative if r is non-negative, *including* the case $\sigma = 0$.
(ii) $b(r)$ has an attractive simple zero at some finite positive value r^*;
(iii) all derivatives of $b(r)$ and $a(r)$ exist for $0 < r < \infty$;
(iv) $a(0) = 0$, and $a(r) > 0$ for $r > 0$;
(v) the risk-adjusted equilibrium probability density $p(r)$, where

$$p(r) = \frac{C}{a(r)} \exp\left\{\frac{2}{\sigma^2}\int [b(r)/a(r)]dr\right\}$$

exists and may be normalized (i.e., $\int_0^\infty p(r)dr < \infty$), where C is the normalization constant[2].

Each time we use one of these assumptions (i)-(v), we make note of it. . Certain additional assumptions are needed for our application of the calculus of variations: we mention them later. These assumptions are satisfied by most of the special case models in the literature and generalize well to multi-factor models as we discuss in Section 4.

Previous theoretical work on eq. (2) has stressed exactly solvable models or certain numerical approaches. Well known among the numerical approaches are the Monte Carlo method and grid-based or binomial model numerical solutions to eq. (2) [see Nelson and Ramaswamy (1990) for the binomial model method]. The Monte Carlo method, for example, exploits the fact that [under certain assumptions: see Duffie (1992)] there is a generalized Feynman-Kac solution [see Kac (1949)] to eq. (2) given by

$$P(r, \tau) = E_r\left\{\exp[-\int_0^\tau x(s)ds]\right\}. \qquad (3)$$

Here $E_r\{...\}$ denotes the expectation with respect to the event space of all possible risk adjusted random walks generated by $dx(s) = b(x(s))ds + \sigma\sqrt{a(x(s))}dz(s)$ with $x(0) = r$. This random process is easily simulated in a discrete approximation to provide estimates for eq. (3). The Monte Carlo approach can be a useful approximation technique, although the method converges very slowly. For our purposes, the Feynman-Kac solution is a good starting place to discuss the regime of small volatility; this occurs in Section 1. In Section

2, we explore the small time to maturity limit. In Section 3, we consider the large time to maturity limit. In that section we derive eq. (1) and a general recursion system to all orders in volatility for the combined regime of large time and small volatility. For the case of large time to maturity and arbitrary volatility, we present a variational approach. We conclude with some brief comments on how each expansion generalizes to multi-factor models and a summary.

1. The Limit of Small Volatility.

1.1 Development of the Expansion.

In this section, we discuss how to develop a power series expansion for the discount bond price in powers of the volatility parameter σ^2. That is, we are going to solve eq. (2) with an expansion, for $N = 1,2,3,\ldots$,

$$P(r,\tau) = \sum_{n=0}^{N-1} \sigma^{2n} P^{(n)}(r,\tau) \quad + O(\sigma^{2N}), \tag{4}$$

where we use superscripts in parentheses to indicate the order of the approximation. Suppose there is no uncertainty in the rate evolution so that $\sigma = 0$. Then we have the same solution as eq. (3), but with the expectation symbol removed and $x(\tau)$ determined as the solution to the deterministic characteristic problem

$$\frac{dx}{d\tau} = b(x), \qquad x(0) = r.$$

The solution just described is the first approximation $P^{(0)}(r,\tau)$ in the power series expansion of eq. (4). That is, $P^{(0)}(r,\tau) = \exp[-\int_0^\tau x(s)ds]$. This *classical* solution is exact when there is no uncertainty in the short-term rate evolution. At this leading approximation, implied forward rates and expected (and actual) future short-term rates are identical, namely all equal to $x(\tau)$. Since $P^{(0)}(r,0) = 1$, we have the initial conditions $P^{(n)}(r,0) = 0$, $n \geq 1$. Substituting eq. (4) into eq. (2) shows that the subsequent terms are determined by the *first-order* partial differential equation

$$b(r)\frac{\partial P^{(n)}}{\partial r} - \frac{\partial P^{(n)}}{\partial \tau} - rP^{(n)} = H(r,\tau) \equiv -\frac{1}{2}a(r)\frac{\partial^2 P^{(n-1)}}{\partial r^2} \tag{5}$$

The notation emphasizes that the previously determined term on the right-hand side of eq. (5) merely acts as a source term $H(r,\tau)$ for the first-order equation at the next order.

Now, first-order partial differential equations like eq. (5) are readily solved. The reader can easily check, for example, that the solution to the slightly more general first-order problem

$$b(x)\frac{\partial f}{\partial x}-\frac{\partial f}{\partial \tau}+c(x)f=h(x,\tau),\qquad f(x,0)=\phi(x), \tag{6}$$

is given by

$$f(x,\tau)=\phi(X(\tau))\,\exp[\int_0^{\tau}c(X(s))ds\,]$$

$$-\int_0^{\tau}\left\{h(X(\tau-s),s)\,\exp\left[\int_s^{\tau}c(X(\tau-\lambda))d\lambda\,\right]\right\}ds \tag{7}$$

where $X(\tau)$ is the solution to $dX/d\tau=b(X)$, $X(0)=x$. Obviously, $X(\tau)$ is also a function of the initial value x, but for notational simplicity we suppress this dependence. It is this dependence, however, through which the right-hand-side of eq. (7) becomes a function of x. Applying eqs. (6) and (7) directly to eq. (5) and using the initial value condition $P^{(n)}(r,0)=0$, $n\geq 1$ gives the solution:

$$P^{(n)}(r,\tau)=\tfrac{1}{2}\int_0^{\tau}\left\{a(x(\tau-s))\frac{\partial^2 P^{(n-1)}(x(\tau-s),s)}{\partial r^2}\exp[-\int_s^{\tau}x(\tau-\lambda)d\lambda]\,\right\}ds \tag{8}$$

1.2 An Example.

Suppose $a(r)=r^{2\theta}$, and $b(r)=Ar-Br^2$. The solution to the characteristic system $dx/d\tau=b(x)$, $x(0)=r$ is readily found to be $x(\tau)=r\exp(A\tau)\xi(r,\tau)$, using $\xi(r,\tau)\equiv A/\{Br[\exp(A\tau)-1]+A\}$. As long as $r>0$, $x(\tau)$ is driven with increasing time to maturity to the fixed point at the zero of $b(r)$, in this case A/B. In fact, one might think of $x(\tau)$ as the "-1 th " approximation to the yield curve, since the "0 th" approximation is simply the time average of $x(\tau)$:

$$P^{(0)}(r,\tau)=\exp[-\int_0^{\tau}x(s)ds]=\xi(r,\tau)^{1/B}.$$

Beyond this order, the various further integrations can be done numerically or, for special values of θ, analytically. For example, if $\theta=3/2$, then eq. (8) yields for the next order of approximation:

$$P^{(1)}(r,\tau)=-(\frac{1+B}{2B^3})\xi^{1/B}\left[\ln\xi+\tfrac{1}{2}(3-\xi)(1-\xi)\right].$$

The yield curve, of course, at the "n th" approximation is obtained by taking $(-1/\tau)$ times the natural log of the partial sum version of eq. (4) to n terms.

1.3 An Exact Solution and a Numerical Comparison.

When $\theta = 3/2$, the example model $dr = (Ar - Br^2)dt + \sigma r^{3/2} dz$ has an exact solution[3] so we can compare various orders of approximation to the known answer. First, define three constants

$$\mu = (1/2) + (B/\sigma^2), \qquad \alpha = -\mu + \sqrt{\mu^2 + (2/\sigma^2)}, \quad \text{and} \qquad \beta = 1 + 2\sqrt{\mu^2 + (2/\sigma^2)}.$$

Next, define the variable $z = z(r,\tau) \equiv 2A/[\sigma^2 r(e^{A\tau} - 1)]$. Finally, the exact solution is given by

$$P(r,\tau) = \frac{\Gamma(\beta-\alpha)}{\Gamma(\beta)} z^{\alpha} M(\alpha,\beta,-z), \tag{9}$$

where $\Gamma(...)$ is the Gamma function and $M(\alpha,\beta,-z)$ is the confluent hypergeometric function [see Erdelyi (1953) for properties]. Comparisons to the series results are reported in Table 1 for a typical set of parameter values and over various maturities. As one sees, the first few orders of the expansion can produce yields accurate to within 1/10 of a basis point for bonds with maturities out to roughly 15 years. The accuracy declines with increasing time to maturity and volatility as one would expect.

1.4 More Examples.

The above case is typical of this particular expansion in that one can usually compute the first two terms analytically; beyond that, a numerical evaluation may be required. The analytical results for a number of simple models are presented in Table 2. Some of these models have exact solutions and some do not. If one wants to investigate a new model, the first step may be to find an exact solution. If this fails, it may be useful to develop the first two terms of the volatility expansion before turning to numerical approaches. For applications where one wants to graph the entire yield curve, for example, under various changing parameter values, the first two or three terms of the volatility series may provide a sufficiently accurate approximation. This will probably be much easier than trying to repeatedly solve the differential equation numerically and more accurate than the slowly convergent Monte Carlo method. The results in Table 2 may look complicated; however, much simplification occurs when the time to maturity is either very small or very large. These regimes are explained in the next two sections.

2. The Limit of Small Time to Maturity.

2.1 Development of the Expansion.

The behavior of the yield curve at the short maturity end gives important clues to the specification of the risk adjusted rate of drift $b(r)$ and the variance rate $a(r)$. How to

analyze this limit is explained in this section.

At the short end, the way to proceed is to think of eq. (2) in the more abstract form $-\mathcal{A}P = \partial P/\partial\tau$, where $\mathcal{A}$ is the linear differential operator

$$\mathcal{A} = -\frac{1}{2}\sigma^2 a(r)\left(\frac{d^2}{dr^2}\right) - b(r)\left(\frac{d}{dr}\right) + r.$$

The minus sign is for notational convenience later. Then, eq. (2) has the asymptotic expansion, valid for $N = 1,2,3,...$

$$P(r,\tau) = \exp(-\mathcal{A}\tau)\, P(r,0) = \sum_{n=0}^{N-1} \frac{(-\tau\mathcal{A})^n}{n!} P(r,0) + O(\tau^N) \qquad (10)$$

For the discount bond, we have $P(r,0) = 1$, but the expansion holds for smooth enough generalized payoffs $P(r,0)$ also. Given a function $P(r,\tau)$ and an expansion parameter τ, the asymptotic expansion is unique. However, the converse is not true: a given asymptotic expansion does not determine a function uniquely. With $P(r,0) = 1$, the low order terms of the series are very easily computed. For example,
$\mathcal{A}\,1 = r$,

$$\mathcal{A}^2\,1 = \mathcal{A}\,r = r^2 - b(r),$$

and

$$\mathcal{A}^3\,1 = r^3 + b(r)[b'(r) - 3r] + \frac{\sigma^2}{2} a(r)[b''(r) - 2],$$

where the primes refer to differentiation with respect to r. As one sees, each higher power of τ comes with various higher derivatives of $a(r)$ and $b(r)$; by assumption (iii) of the Introduction , all these derivatives exist. Hence

$$P(r,\tau) = \exp[-R(r,\tau)\tau]$$
$$= \exp\left(-r\tau - b(r)\frac{\tau^2}{2} - \left\{2b(r)b'(r) + \sigma^2 a(r)[b''(r) - 2]\right\}\frac{\tau^3}{12} + O(\tau^4) \right) \qquad (11)$$

Next we present examples that illustrate the nature of the series in particular cases.

2.2 Examples.

Ex (2.1)

Consider the CIR $\theta = 1/2$ model defined by $a(r) = r$, $b(r) = A - Br$. Define the constant $\delta = (B^2 + 2\sigma^2)^{1/2}$ and three functions

$$\gamma(\tau) = (\delta + B)\exp(\delta\tau) + (\delta - B),$$

$$\psi(\tau) = \left(\frac{2\delta}{\gamma(\tau)}\right)^{2A/\sigma^2} \exp\left[\frac{(\delta + B)A\tau}{\sigma^2}\right], \quad \text{and}$$

$$\varphi(\tau)=\frac{2}{\gamma(\tau)}[\exp(\delta\tau)-1].$$

Then, the exact solution (see Feller[1951]/CIR[1985]) is given by $P(r,\tau)=\psi(\tau)\exp[-\varphi(\tau)r]$. Expanding the exact solution $P(r,\tau)$ in powers of τ and exponentiating gives

$$P(r,\tau)=\exp\left\{-r\tau-(A-Br)\frac{\tau^2}{2}+[(A-Br)B-r\sigma^2]\frac{\tau^3}{6}+O(\tau^4)\right\},$$

which is easily seen to be in agreement with the eq. (11) expansion. In this case, one sees that the expansion is convergent and converges to the exact solution. In *contrast*, consider:

Ex (2.2)

Consider again the equation (9), $\theta=3/2$ model defined by $a(r)=r^3$, $b(r)=Ar-Br^2$. In the limit $\tau\to 0$, we have the real variable $z(r,\tau)\to+\infty$. Hence, we need to apply the asymptotic formula [see Erdelyi (1953)], valid as $z\to+\infty$, for $N,M=1,2,3,\ldots$,

$$M(\alpha,\beta,-z)=\frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}z^{-\alpha}\sum_{n=0}^{N-1}\frac{(\alpha)_n(\alpha-\beta+1)_n}{n!}z^{-n}+O(|z|^{-\alpha-N})$$

$$+\frac{\Gamma(\beta)}{\Gamma(\alpha)}(-z)^{\alpha-\beta}e^{-z}\sum_{n=0}^{M-1}\frac{(\beta-\alpha)_n(1-\alpha)_n}{n!}(-z)^{-n}+O(|e^{-z}z^{\alpha-\beta-M}|). \qquad (12)$$

The notation $(\alpha)_n$ is defined by the relations $(\alpha)_0=1$, and $(\alpha)_n=\alpha(\alpha+1)\cdots(\alpha+n-1)$, $(n=1,2,3,\ldots)$. The sums are asymptotic expansions and clearly not convergent since for large n, $(\alpha)_n$ grows like $n!$. The first summation for $M(\alpha,\beta,-z)$, and using eq. (9), implies that

$$P(r,\tau)=\sum_{n=0}^{N-1}\frac{(\alpha)_n(\alpha-\beta+1)_n}{n!}z(r,\tau)^{-n}+O(|z(r,\tau)|^{-N}). \qquad (13)$$

Expanding $z(r,\tau)^{-1}=(\sigma^2 r/2)(\tau+A\tau^2/2+A^2\tau^3/6)+O(\tau^4)$ and substituting into the equation directly above yields

$$P(r,\tau)=\exp\left\{-r\tau-(Ar-Br^2)\frac{\tau^2}{2}+[-A^2r+3ABr^2-2B^2r^3+\sigma^2(1+B)r^3]\frac{\tau^3}{6}+O(\tau^4)\right\}$$

which is in agreement with eq. (11). The second summation in eq. (12) for $M(\alpha,\beta,-z)$, because of the leading factor $\exp(-z) \approx \exp(-2/r\sigma^2\ \tau)$, consists entirely of terms that vanish faster than any power of τ. As we will see, this second sum can often be neglected with no practical error if the time to maturity is small enough. So, in this example, we see that the eq. (10) expansion is merely asymptotic and not convergent. Since an asymptotic sum does not determine the function even when it converges[4], the reader should not, in general, expect to be able to "sum" the series and then extrapolate to another regime such as $\tau \to \infty$. Nevertheless, the numerical performance of the expansion, when truncated, can be very accurate and closely mimic that of a convergent series until N becomes very large. For example, take the parameter values of Table 1, namely $A = .024, B = .4, \sigma = .2, r = .03$ and suppose $\tau = 36$ months so that $z = 29.1411$. Then, the partial sums of eq. (13) for $P(r,\tau)$ have a "pseudo-convergence" to the value .274668... by $N = 20$ which is graphed in Figure 1. More specifically, the partial sums remain at the value .274668... for $N = 20, 21, 22, \ldots, 110$, change in the sixth decimal place at $N = 111$, and then exponentially diverge for higher values of N. So, for an extremely lengthy sequence of partial sums, the series behaves just like a convergent one, with an associated value for $R(r,\tau) = (-1/36)\ln(.274668...) = 3.5894...\%$ which is equal to the exact value to the precision reported. Since, for practical applications, a yield accurate to 4 significant figures is usually sufficient, this example and further examples below suggest that these particular expansions can be treated "as if" they were convergent for numerical estimates when the time is small enough.

The numerical performance of the small time expansion is further illustrated below under the section "2.5 Numerical Testing".

2.3 The Initial Slope and Curvature of the Yield Curve.

We see from eq. (11) that $b(r)$ does not affect the price of the short maturity discount bond until $O(\tau^2)$; this fact helps justify the time series empirical work on T-bills as a measurement of $m(r)$ and not $b(r)$. (This $O(\tau^2)$ argument can also be made in the presence of additional explanatory factors in a multi-factor model; see the discussion in Section 4.1). However, we see that the initial slope of the yield curve is a measurement of $(1/2)b(r)$, being positive or negative depending on which side of the attractor one happens to find the short-term rate at the time of measurement. Since $b(r) = m(r) + \sigma\lambda(r)\sqrt{a(r)}$, and $\sigma\sqrt{a(r)} > 0$ it will be the sign of $\lambda(r)$ that controls the relatinship between the zero crossing of $b(r)$ and $m(r)$. For example, when the risk adjustment is positive, the yield curve will initially be flat at a rate $r^* > r^{**}$, where r^{**} is the zero of $m(r)$. This relationship is illustrated schematically in Figure 2. As one sees from eq. (11), the first influence of the rate variance $a(r)$, apart from its contribution to the risk adjustment in $b(r)$, shows up as a part of the initial curvature of the yield curve..

So, in theory, $a(r)$ and $b(r)$ are determined over the range (r_1, r_2) from measurements on the initial slope and curvature of the yield curve during the time that r

varies between r_1 and r_2. Moreover, by watching long enough, r_1 can be taken to arbitrarily small (strictly positive) values and r_2 can be taken to arbitrarily large (finite) values. This is because, by our assumptions that the equilibrium density exists and yet $a(r)$ never vanishes for positive r, every finite strictly positive interest rate will eventually be reached by the risk-adjusted Brownian motion [see Feller (1954) and the references therein for proofs].

2.4 The General Structure of the Small Time Expansion.

The two limits $\sigma^2 \to 0$ and $\tau \to 0$ are distinct. For example, one can let $\tau \to \infty$ in each term in the volatility expansion and the result will exist; however, if one truncates the time to maturity expansion at a finite number of terms, it does not make sense to then let $\tau \to \infty$. Nevertheless, at a fixed order τ^m in the time to maturity expansion, only powers of σ^2 up to and including $(\sigma^2)^{m-1}$ can appear. That is, the general structure of the τ expansion for the yield to maturity is

$$R(r,\tau) = R_{0,0} + \tau R_{1,0} + \tau^2 (R_{2,0} + \sigma^2 R_{2,1}) + \tau^3 (R_{3,0} + \sigma^2 R_{3,1} + \sigma^4 R_{3,2}) + \ldots$$

where $R_{i,j}$ denotes the coefficient of $\tau^i \sigma^{2j}$ and depends on the drift and volatility specifications $b(r)$ and $a(r)$. For the general case of arbitrary drift and volatility functions, these first few terms are

$$R_{0,0} = r, \qquad R_{1,0} = \frac{1}{2} b(r), \qquad R_{2,0} = \frac{1}{6} b(r) b'(r),$$

$$R_{2,1} = \frac{1}{12} a(r)[b''(r) - 2], \qquad R_{3,0} = \frac{1}{24}[b(b')^2 + b^2 b''],$$

$$R_{3,1} = \frac{1}{48}[ba'(b'' - 2) + a(2b''' + 3b'b'' - 6b')],$$

$$\text{and} \quad R_{3,2} = \frac{1}{96} a[ab^{iv} + 2a'b''' + a''(b'' - 2)].$$

2.5 Numerical Testing.

How does the series perform numerically? It is not as efficient as the volatility expansion, and requires more terms for a given accuracy, but this is offset by the fact that the individual terms are easier to compute. Tables 3 through 6 show the performance of the series through various orders for the risk-adjusted processes $dr = (A - Br)dt + \sigma r dz$ and $dr = (Ar - Br^2)dt + \sigma r^{3/2} dz$. In the latter case, the same parameters are used as in Table 1 to permit a comparison. In the former case, A, B, and σ^2 are given the Table 1 values times the short-term rate (.03 in the example). This has the effect of "rescaling" the parameters to the appropriate order of magnitude for that particular model. Also, because

$R_{1,0} = (1/2)b(r)$, this change keeps the *first-order* approximation identical in all the examples as one can see in the tables. The criteria used to construct the "final estimate" at the bottom of each table was (i) once a yield repeated to 4 significant figures, "convergence" to that accuracy was assumed, and (ii) in any event, no more than 8 orders (i.e. $R(r,\tau)$ through the τ^8 term) of the expansion were calculated. For the simple models tried, one can see that 4 significant figure accuracy was obtained through about 5 years to maturity with a fairly rapid loss of precision beyond that time limit. In particular, Tables 5 and 6 show that even though we know the series is only asymptotic for that particular model, the truncated series can be accurate enough for applications. Suppose one is working with a model in which the drift and volatility are easily differentiated and the time to maturity of interest is under 5 years. Then, the time to maturity expansion is clearly a serious alternative to a grid-based numerical solutions for 4 digit precision, which is sufficient for most applications.

3. The Limit of Large Time to Maturity.

3.1 The Eigenvalue Problem

Next, consider the long maturity end of the yield curve. One of the most characteristic features of the yield curve is its tendency to flatten to a limit as the time to maturity increases. Of course, in the U.S. we really only have experience of long term maturities in the 30 to 90 year range, but the general tendency of yield curves to flatten is apparent at much smaller maturities. Most observers would concede the existence, at any particular point in time, of an idealized limiting yield to maturity which we shall denote as $R(r,\infty)$ in stationary models, and defined by

$$R(r,\infty) \equiv \lim_{\tau\to\infty} \frac{-1}{\tau} \ln P(r,\tau).$$

One of the simplest ways for this limit to exist is for $P(r,\tau)$ to have its dependence on r and τ *separate* in the limit of large time to the form $P(r,\tau) \to u(r)\exp(-\lambda_0 \tau)$, where λ_0 is a constant independent of both r and τ. Of course, in this separable case, we conclude that $R(r,\infty)$ is independent of r, which we denote by writing

$$R_\infty = R(r,\infty) = \lambda_0.$$

In fact, in the types of models that we are examining here, this is exactly what happens. We show this by construction below and thus justify the separation assumption. Our construction is most easily carried out in the combined regime of large time to maturity and small volatility. But, after we complete our construction and discuss various examples, we then consider the regime of large time to maturity and arbitrary volatility.

The reason we write λ_0 instead of R_∞ is that λ_0 is a more conventional notation for an eigenvalue. That is, after substituting $P(r,\tau) = u(r)\exp(-\lambda_0 \tau)$ in eq.

(2), we obtain $\mathcal{A}u = \lambda_0 u$. Not only is λ_0 an eigenvalue for the operator $\mathcal{A}$ and $u(r)$ the associated eigenvector, but in fact λ_0 is the smallest such eigenvalue (i.e., the principal or fundamental eigenvalue).

More generally, second order partial differential equations related to our type have been extensively studied. Under very general conditions [see Titchmarsh (1962)], we can expect $P(r,\tau)$ to have an *eigenfunction* expansion of the form:

$$P(r,\tau) = \sum_{\lambda_n \in p(\mathcal{A})} f_n u_n(r) \exp(-\lambda_n \tau) + \int_{\lambda \in c(\mathcal{A})} f(\lambda) u_\lambda(r) \exp(-\lambda\tau) d\lambda. , \qquad (14)$$

where the f_n and $f(\lambda)$ are expansion coefficients. The set of values $\{\lambda\}$ which contribute to this expansion is called the spectrum of the operator $\mathcal{A}$. The sum ranges over the point or discrete spectrum $p(\mathcal{A})$ and the integral ranges over the continuous spectrum $c(\mathcal{A})$. Special cases include that of a pure point spectrum or a pure continuous spectrum. But, in general, there will be a mixture of both. When this occurs, the typical situation is that the continuous spectrum occupies values on the λ-axis greater than the point spectrum values. One can see from the expression directly above that the *second* contribution to the large time limit is at least a factor $\exp[-(\lambda_1 - \lambda_0)\tau]$ smaller than the leading term, where λ_1 is either the second smallest discrete eigenvalue or the smallest value in the continuous spectrum.

In fact, for each of the three exactly solvable models discussed in Examples 2.1, 2.2 and one more below, all of the terms of the eigenfunction expansion can be developed, although we shall not do so here. Two of the models have a purely discrete spectrum and one has both a discrete and continuous spectrum. In this paper, we want to work with just the leading term of the large time behavior. So, for notational simplicity, and because we are only going to discuss the leading term of eq. (14), we will drop the *subscripts* in λ_0 and u_0. From this point on, λ will denote the principal eigenvalue and $u(r)$ shall denote the associated eigenfunction. Also, we are going to employ *superscripts* as in Section 1 to denote terms in a volatility expansion, so our notation will save us from having to write both subscripts and superscripts at the same time.

In general, the eigenvalues associated with a differential operator depend on the *domain* of the operator: that is, the space of functions on which it is defined, which includes either boundary conditions or integrability conditions. In our case, as will be apparent in the construction below that leads to eq. (1), we have *no* boundary conditions either at $r = 0$ or $r = \infty$. This can make mathematical sense when and only when these boundary points are singular points; that is why we assume $a(r)$ vanishes at $r = 0$ (assumption (iv) of the Introduction). Similarly, we assume the point at infinity is singular also. The latter singularity is always generated by the $-rP$ term in eq. (2) and sometimes the other terms contribute additional singularities at $r = \infty$.

A general theory of the Sturm-Liouville form equation $\mathcal{L}f = \lambda\rho f$ with singular endpoints has been developed by Weyl, where $\mathcal{L}f = -(\xi(r)f')' + q(r)f$. One requires that $\xi(r) > 0, \rho(r) > 0$ away from the singular endpoints. Our seemingly more general problem $\mathcal{A}u = \lambda u$ can be transformed into the Sturm-Liouville form by a substitution of the form $u(r) = g(r)f(r)$, where $g(r) = \rho(r) = 1/p(r) > 0$ and $\xi(r) = (1/2)\sigma^2 a(r)/p(r) > 0$, if $0 < r < \infty$. When no (general homogeneous) boundary conditions can be imposed, Weyl first proved [see Hille (1969)] that this corresponds to the existence (for complex values for λ such that $\mathrm{Im}(\lambda) \neq 0$) of two unique solutions to $\mathcal{L}f = \lambda\rho f$. These unique solutions $u_{1,2}$ fall off rapidly enough toward the endpoints so that certain integrability conditions hold. If we translate these conditions back to our problem, they become:

$$\int_0^\kappa p(r)|u_1(r)|^2\,dr < \infty, \quad \text{and} \quad \int_\kappa^\infty p(r)|u_2(r)|^2\,dr < \infty$$

where κ is an arbitrary, but interior point, $0 < \kappa < \infty$ and $|u(r)|^2 = \bar{u}(r)u(r)$. Here $\bar{u}(r)$ denotes the complex conjugate of $u(r)$. Since the solutions are unique, one cannot specify general boundary conditions at the endpoints; this inability is called by Weyl the "limit point" case. In contrast, when boundary conditions can be specified, this is called the "limit circle" case. Thus, our assumption of no boundary conditions at $r = 0$ and $r = \infty$ can be rephrased as an assumption that both of these points are in the limit point case. [Again, see Hille (1969) for elaboration]. We discuss the integrability conditions appropriate for our problem below.

A related, but probabilistic approach to these questions, is the characterization of the boundaries by Feller as *accessible* or *inaccessible* to the risk-adjusted Brownian motion [see Feller (1954)], in the sense that the probability that the boundaries will be reached in finite time is strictly positive or zero, respectively. In Feller's theory, the boundaries are classified as *natural, entrance, exit,* or *regular.* There is not a direct correspondence between Feller's classifications and Weyl's because of the behavior of the term $-rP$ in eq. (2). That is, this particular term causes singular behavior as $r \to \infty$ in Weyl's theory, but does not have any effect on Feller's classifications for the risk-adjusted Brownian motion.

At this point in our argument, what can we say about the possible values for λ? From eq. (3) it is obvious that $0 \leq P \leq 1$ since we assume that $x(s) \geq 0$ (assumption (i) of the Introduction) ; hence $\lambda \geq 0$. Again from eq. (3) and Jensen's Inequality [see Beaglehole and Tenney (1992)] $R(r,\tau) \leq (1/\tau)E_r[\int_0^\tau x(s)ds]$. If one lets $\tau \to \infty$ in this last inequality and uses the assumption that an equilibrium exists (assumption (v) of the Introduction), one achieves the upper bound $\lambda \leq E[r] \equiv \int_0^\infty rp(r)dr$. This last expression on the right hand side is the risk-adjusted equilibrium expectation (using $E[\ldots]$) for the short rate. In fact, this particular bound is very poor and we construct the best possible

upper bound in section "3.4 Large Time to Maturity and Arbitrary Volatility: Variational Principle Bounds".

3.2 Large time to maturity and small volatility.

With these preliminaries out of the way, we now specialize to the combined regime of large time to maturity and small volatility. In this double limit, we can calculate λ as a formal power series in σ^2. This will confirm, by construction, the postulated large time behavior $P(r,\tau) \to u(r)\exp(-\lambda\tau)$ to $O(\sigma^{2N})$, where N is an arbitrary integer.

Note that the above expression for Jensen's Inequality becomes exact if $\sigma^2 = 0$. Hence, to the first approximation and using the same superscript notation as in Section 1,

$$\lambda = \lambda^{(0)} = \lim_{\tau\to\infty} \frac{1}{\tau}\int_0^\tau x(s)dx,$$

where $x(s)$ is the deterministic solution to $dx/ds = b(x), x(0) = r$. But, $\lim_{s\to\infty} x(s) = r^*$ by our fixed point assumption (ii) of the Introduction; this demonstrates that $\lambda^{(0)} = r^*$. Then, the associated eigenfunction $u(r) = u^{(0)}(r)$ is given to the same order of approximation as the solution to

$$b(r)\frac{du^{(0)}}{dr} - r\,u^{(0)} = -\lambda^{(0)}u^{(0)} = -r^*u^{(0)}$$

which is immediately integrated to give

$$u^{(0)}(r) = \exp\left\{\int_{c_0}^{r} [\frac{x-\lambda^{(0)}}{b(x)}]dx\right\} \qquad (15)$$

where c_0 is a normalization point. We will enforce the normalization that $u(c_0) = 1$ to all orders in σ^2. Note that eq. (15) confirms our choice for $\lambda^{(0)}$ because (i) since $\partial P/\partial r$ exists (by assumption), du/dr exists; (ii) du/dr exists at $r = r^*$ only if $\lambda^{(0)} = r^*$. This argument also requires that the zero for $b(r)$ must be simple at r^* (assumption (ii) of the Introduction). In fact, the normalization point $c_0 = r^*$ as may be seen as follows. If we generalize to a payoff function $P(r,0) = \phi(r)$ not necessarily a constant, then eq. (7) yields the slightly more general

$$P^{(0)}(r,\tau) = \phi(x(\tau))\exp[-\int_0^\tau x(s)ds] = \phi(x(\tau))\exp\left\{-r^*\tau - \int_0^\tau [x(s)-r^*]ds\right\}.$$

But, we can change integration variable from s to $x(s)$, using $ds = dx/b(x)$, yielding

$$P^{(0)}(r,\tau) = \phi(x(\tau))\,\exp\left\{-r^*\tau + \int_{x(\tau)}^{r}\left[\frac{x-r^*}{b(x)}\right]dx\right\}.$$

Finally, taking τ large again confirms eq. (15), but also shows that $c_0 = r^*$, and provides the additional (leading order) multiplicative normalization constant $\phi(r^*)$ in the case of a generalized payoff function. (Note: beyond leading order our normalization may differ from what one would see by taking the $\tau \to \infty$ limit of $P(r,\tau)$; this is irrelevant to the calculation of λ.)

Proceeding to higher orders, we write, for $N = 0, 1, 2, \ldots$

$$\lambda = \lambda^{(0)} + \sigma^2 \lambda^{(1)} + \ldots + O(\sigma^{2N}),$$

$$u(r) = u^{(0)}(r) + \sigma^2 u^{(1)}(r) + \ldots + O(\sigma^{2N}),$$

and substitute into $\mathcal{A}u = \lambda u$, collecting terms of the same order. We have $u^{(1)}$ determined by

$$\frac{du^{(1)}}{dr} - \frac{(r-r^*)}{b(r)}u^{(1)} = h^{(1)}(r) \equiv \frac{-1}{b(r)}\left[\lambda^{(1)}u^{(0)} + \tfrac{1}{2}a(r)\frac{d^2u^{(0)}}{dr^2}\right],$$

with solution

$$u^{(1)}(r) = \int_{r^*}^{r}\left(h^{(1)}(x)\exp\left\{\int_x^r\left[\frac{z-r^*}{b(z)}\right]dz\right\}\right)dx.$$

Using eq. (15),

$$h^{(1)}(r) = -\left\{\lambda^{(1)} + a(r)\frac{[(r-r^*)^2 - (r-r^*)b'(r) + b(r)]}{2[b(r)]^2}\right\}\frac{u^{(0)}(r)}{b(r)}.$$

But du/dr will again fail to exist unless the expression in braces just above vanishes as $r \to r^*$. An application of L'Hospital's rule then gives us through second order

$$\lambda = r^* - \sigma^2\,\frac{a(r^*)}{4[b'(r^*)]^2}[2 - b''(r^*)] + O(\sigma^4). \qquad (16)$$

A general recursion relation is as follows. At order $n \geq 1$, $u^{(n)}(r)$ is determined from

$$u^{(n)}(r) = \int_{r^*}^{r} \left(h^{(n)}(x) \exp\{\int_x^r [\frac{z-r^*}{b(z)}]dz\} \right) dx \qquad (17)$$

where

$$h^{(n)}(x) = \frac{-1}{b(x)}\left[\lambda^{(n)} u^{(0)}(x) + \lambda^{(n-1)} u^{(1)}(x) + \ldots + \lambda^{(1)} u^{(n-1)}(x) + \frac{a(x)}{2}\frac{d^2 u^{(n-1)}(x)}{dx^2}\right]$$

. (18)

As before, $h^{(n)}(x)$ must exist at $x = r^*$, which means that the expression just above in brackets must vanish at $x = r^*$. For most of the terms, this will happen automatically because $u^{(i)}(x = r^*) = 0$, for $i \geq 1$ using our normalization. But since $u^{(0)}(x = r^*) = 1$, we see that $\lambda^{(n)}$ is determined by

$$\lambda^{(n)} = -\frac{1}{2} a(r^*) \frac{d^2 u^{(n-1)}(r^*)}{dx^2} \qquad (19)$$

For $n = 1$, eq. (19) is easily verified to agree with eq. (16).

3.3 Four Examples.

Ex (3.1) Consider again the Feller (1951)/CIR (1985) $\theta = 1/2$ model defined by $a(r) = r$ and $b(r) = A - Br$. Hence, from eq. (16), $\lambda = (A/B) - \sigma^2 A/(2B^3) + O(\sigma^4)$. From the known exact solution: $\lambda = (A/\sigma^2)[(B^2 + 2\sigma^2)^{1/2} - B]$; expanding the exact solution in powers of σ^2 shows agreement with eq. (16). Also note the exact lowest eigenfunction is $u(r) = \exp(-\lambda r/A)$. The steady state density $p(r) = Cr^{-1+2A/\sigma^2}\exp(-2Br/\sigma^2)$ is normalizable for all $A > 0, B > 0$. The entire spectrum is discrete.

Ex (3.2) Consider again the $\theta = 3/2$ model defined by $a(r) = r^3$ and $b(r) = Ar - Br^2$. One can show the exact result $\lambda = (A/\sigma^2)\{[(B + \sigma^2/2)^2 + 2\sigma^2]^{1/2} - (B + \sigma^2/2)\} = (A/B) - \sigma^2 A(1+B)/(2B^3) + O(\sigma^4)$, again in agreement with eq. (16). In this case the associated eigenfunction is $u(r) = r^{-\lambda/A}$. The steady state density $p(r) = Cr^{-3-2B/\sigma^2}\exp(-2A/r\sigma^2)$, normalizable for all $A > 0, B > -\sigma^2$. The entire spectrum is discrete.

Ex (3.3) Consider the $\theta = 1$ (*first logarithmic*) model defined by $a(r) = r^2$ and $b(r) = Ar - Br^2$. [See Merton (1975) for an equilibrium limit application]. One can demonstrate the exact results[5] $\lambda = (A/B) - \sigma^2(1+B)/(2B^2)$ for $\sigma^2 \leq 2AB/(B+2)$, with

the associated eigenvector $u(r) = r^{-1/B}$. This is the entire range of σ^2 for which there exists a point spectrum. However, λ continues to exist as a strictly positive eigenvalue at the bottom of the continuous spectrum when σ^2 is in the range $2AB/(B+2) \le \sigma^2 \le 2A$, in which case it can be shown that $\lambda = \delta \equiv (2A - \sigma^2)^2/8\sigma^2$. In this last parameter range, the entire spectrum is purely continuous, extending over the entire interval $[\delta, \infty)$. To summarize: the exact $\lambda(\sigma^2)$ is a continuous, decreasing function of σ^2, reaching zero at $\sigma^2 = 2A$. At this point, the equilibrium density $p(r)$ no longer can be normalized. In fact, $p(r) = Cr^{-2+2A/\sigma^2}\exp(-2Br/\sigma^2)$, normalizable for all $\sigma^2 < 2A$, $B > 0$. So in this remarkable case with both a discrete and continuous spectrum, the first two terms of eq.(16) are exact for a non-zero region about $\sigma^2 = 0$.

Ex. (3.4) Consider the $\theta = 1$ (*second logarithmic*) model defined by $a(r) = r^2$ and $b(r) = A - Br$. [See Brennan and Schwartz (1980), and Courtadon (1982) for applications of this model]. The steady state density is $p(r) = Cr^{-2-2B/\sigma^2}\exp(-2A/r\sigma^2)$, normalizable for all $A > 0, B > -\sigma^2/2$. The previous three models, all with known solutions for λ and different types of spectra, confirmed the validity of our expansion. This model is the first example of the four in this section where no exact solution is known. Hence, it is interesting in this case to compute several additional terms in the expansion and see how it behaves. The recursion system eqs. (17),(18), and (19) can be automated through several orders with a symbolic algebra system. Using one, we find

$$\lambda = \frac{A}{B} - \sigma^2\frac{A^2}{2B^4} + \sigma^4\left(\frac{A^3}{B^7} - \frac{A^2}{4B^5}\right) + \sigma^6\left[\frac{A^2}{8B^{10}}(-24A^2 + 12AB - B^4)\right]$$

$$+\sigma^8\left[\frac{A^2}{16B^{13}}(176A^3 - 137A^2B^2 + 28AB^4 - B^6)\right]$$

$$-\sigma^{10}\left[\frac{A^2}{32B^{16}}(1456A^4 - 1580A^3B^2 + 550A^2B^4 - 60AB^6 + B^8)\right]$$

$$+O(\sigma^{12}) \qquad (20)$$

With the parameter values of Table 3, the successive partial sums associated with eq. (20) for the yield to maturity, in percent, are: 6.000, 5.625, 5.714, 5.682, 5.696, and 5.689 respectively , suggesting an exact value in the vicinity of 5.69%. On the other hand, when σ is doubled, as in Table 4, the partial sums are erratic with no convergence indicated.

In each of the examples with exact solutions, note that the exact eigenvalue has a Taylor series expansion in $\varepsilon \equiv \sigma^2$, which converges in a region $|\varepsilon| < \Delta$, where we

generalize ε to a complex parameter and $\Delta > 0$. This shows that, in general, our series can be an exact and convergent one for a sufficiently small value of ε. Also, in the one example in which no exact solution was known, the numerical behavior also suggested a non-zero but finite radius of convergence in the expansion parameter σ^2. Of course, as in the case of the small time expansion, in other cases there may be additional terms that vanish faster than any power of the expansion parameter. The general problem of establishing convergence and radius of convergence estimates for the volatility expansion for the eigenvalues is beyond the scope of this article and would merit further research. Standard references on the general topic of convergence proofs for perturbations of linear operators are Rellich (1969) and Kato (1976).

What can we do in the regime where the time to maturity is large and the volatility is also large? For example, *large* means the volatility is outside the radius of convergence of the power series ($|\sigma^2| \geq \Delta$). In this case, we can implement a *variational* approach to determining λ, which is established as follows.

3.4 Large Time to Maturity and Arbitrary Volatility: Variational Principle Bounds.

By multiplying both sides of $\mathcal{A}u = \lambda u$ by $u(r)p(r)$ and integrating by parts, we find that

$$\lambda = \frac{\int_0^\infty p(r)\{\frac{1}{2}\sigma^2 a(r)[u'(r)]^2 + r[u(r)]^2\}dr}{\int_0^\infty p(r)[u(r)]^2 dr}$$

$$= \min_{f(r)} \left(\frac{\int_0^\infty p(r)\{\frac{1}{2}\sigma^2 a(r)[f'(r)]^2 + r[f(r)]^2\} dr}{\int_0^\infty p(r)[f(r)]^2 dr} \right) \qquad (21)$$

under the conditions that the real-valued function $f(r)$ satisfies the endpoint relations:
(i) $\lim_{r \to 0,\infty} a(r)p(r)f(r)f'(r) = 0$, and the further integrability conditions:
(ii) $\int p(r)f^2 dr$, $\int p(r)rf^2 dr$, $\int p(r)a(r)(f')^2 dr$ are convergent. The endpoint behavior and integrability conditions can be verified for the exact eigenfunction on each of the three examples where this function is known. In fact, the prior Examples 3.2 and 3.3 show that $u(r)$ and $u'(r)$ may not, in themselves, actually exist at the boundary at $r = 0$ but diverge there even though the endpoint condition (i) holds. This is in marked contrast to the regular (i.e., non-singular case) theory in a finite interval $r_{\min} \leq r \leq r_{\max}$, where $a(r) > 0$ throughout, including the endpoints. In the regular case, one might specify the

reflecting boundary conditions $u'(r_{\min}) = u'(r_{\max}) = 0$ to create a term structure theory. Of course, the additional specification problem of this regular theory would be the estimation of the values $r_{\min}$ and $r_{\max}$ from policy or other empirical considerations. Note that second identity in eq. (21) follows from the *Euler-Lagrange* equations of the theory of the calculus of variations [see Courant and Hilbert (1989) for discussion and elaboration].

A short-hand way of indicating these integrability conditions is to first define an *inner-product* of two real-valued functions $(f,g) \equiv \int_0^\infty p(r)f(r)g(r)dr$ and a norm $\|f\| \equiv (f,f)^{1/2}$. Then, we can define a Euclidean vector space of functions which we call $L_2^{(p)}$, which is the space of all real-valued functions such that $\|f\| < \infty$. (Technically, a *Hilbert* space if we were more specific about what we mean by *function* and *integral*). Finally, the domain of our operator $\mathcal{A}$, which we denote as $D(\mathcal{A})$, is the set of functions such that both $f \in L_2^{(p)}$ and $\mathcal{A}f \in L_2^{(p)}$.

Equation (21) is a powerful tool that can be used to estimate λ to high accuracy by selecting suitable *trial* functions $f(r)$. Any trial function, if it satisfies the endpoint and integrability conditions, will produce an upper bound estimate. But, more importantly, the relation

$$\lambda \leq \frac{\int_0^\infty p(r)\{\frac{1}{2}\sigma^2 a(r)[f'(r)]^2 + r[f(r)]^2\}\,dr}{\int_0^\infty p(r)[f(r)]^2\,dr} \tag{22}$$

is the tightest possible upper bound because it will be realized as an equality if $f(r)$ is chosen to be the exact eigenfunction. In contrast to this, recall the Jensen Inequality bound $\lambda \leq E[r] \equiv \int p(r)r\,dr$ which was discussed in Section 1. Comparing the Jensen bound with eq. (22), one sees that it is equivalent to selecting the trial function $f(r) = 1$ or any constant. In general, this will be a poor bound, since a constant function will not be the exact eigenfunction.

3.5 A Numerical Example.

Let us continue where we left off in trying to estimate the limiting yield to maturity at large times for the second logarithmic model: $dr = (A - Br)dt + \sigma r dz$. In order to use eq. (22), we must select a trial function that satisfies the endpoint and integrability conditions (i) and (ii) above. The selection of a trial function is somewhat of an art; clearly, one wants to capture the qualitative behavior of the unknown exact eigenfunction. The volatility expansion at the lowest order, namely eq. (15), is very valuable here. It tells us that, as $\sigma^2 \to 0$, $u(r) \to \exp(-r/B)[1 + O(\sigma^2)]$. This suggests the trial function $f(r) = \exp(-\alpha r)$, where $\alpha \geq 0$ is a parameter which is optimized to produce the lowest possible value for the ratio on the right-hand-side of eq. (22). It is easy to verify that this

choice satisfies the endpoint and integrability conditions. Moreover, the two integrals in the numerator of eq. (23) and the one in the denominator are all of the same known form:

$$\int_0^\infty \tau^{\nu-1} \exp(-s\tau - \frac{1}{\tau}) d\tau = 2s^{-\nu/2} K_\nu(2\sqrt{s}),$$

where $K_\nu(...)$ is the modified Bessel function of the second kind of order ν [see Abramowitz and Stegun (1972) for properties.] It is not too surprising that these particular functions appear for this model . The reason is that one can show that for the special case when $A = 0$, one has the exact eigenfunction $u(r) = r^{\nu/2} K_\nu(\sqrt{8r}/\sigma)$, where $\nu = 1 + (2B/\sigma^2)$, and $\lambda = 0$. [see footnote 2 and also Dothan (1978)]. Evaluating the integrals yields

$$\lambda \leq \min_{\omega \geq 0} \left(\frac{1}{K_\nu(\omega)} \left[\frac{\sigma^2 \omega^2 K_{\nu-2}(\omega)}{32} + \frac{4AK_{\nu-1}(\omega)}{\sigma^2 \omega} \right] \right) \qquad (23)$$

where (as in the case $A = 0$) $\nu = 1 + (2B/\sigma^2)$ and $\omega = (4/\sigma)\sqrt{\alpha A}$. Recall that with A, B and σ equal to the values in Table 3, we found an indication from the series eq. (20) above that $\lambda \approx 5.69\%$. When these same values for $A, B,$ and σ are inserted into eq. (23), we find an optimal value for $\omega = 53.8235$ and the resulting bound $\lambda \approx 5.692\%$, so we have consistency. When σ is doubled from this to the Table 4 value, eq. (20) was essentially useless, but eq. (23) yields the optimal $\omega = 24.7103$, with an estimate $\lambda \approx 5.139\%$. These bounds are probably accurate to the number of significant figures reported. The reason for this belief is that one can try to improve the estimates by enlarging the trial function parameter space to, for example,

$$f(r) = (1 + \alpha_2 r)\exp(-\alpha_1 r) \quad \text{or} \quad f(r) = (1 + \alpha_3 r^2)\exp(-\alpha_1 r)$$

and then optimize in the new larger parameter space. Regardless of the value for volatility, the optimal value for α_2 was zero. The addition of the α_3 term contributed to a small lowering of the optimal λ value once σ was raised to about 10% per month. For example, at $\sigma = .10$, the eq. (23) bound was $\lambda = 3.620\%$. Once the α_3 term was added, this lowered the bound to $\lambda = 3.617\%$. This procedure was repeated over a range of volatility from $\sigma = 2\%$ per month to $\sigma = 30\%$ per month; the results are reported in Table 7 and graphed in Figure 3. As one sees, the one-parameter trial function seems to produce yields accurate to 3 significant figures and one is probably picking up one more significant figure with the addition of the second parameter. Given the significant uncertainties in any empirical estimates for the parameters, this may well be all the accuracy one will need for applications.

4. Generalization to Multi-factor Models and Conclusions.

4.1 Generalization.

All of the above techniques can be generalized. For example, the volatility expansion can be applied to multi-factor models. One would require the assumption that the variance-covariance matrix $a_{ij}(x)$ of the n factors x have a common small volatility parameter σ. For example, consider adding an inflation rate factor as in CIR (1985) Model 1 and Model 2. It would be natural that in a world with no inflation uncertainty, there would be no short-term rate uncertainty also. However, in other multi-factor models, this assumption would have to be examined in each case. Assuming such a common volatility parameter, eq. (2) becomes

$$-\mathcal{A}P \equiv \frac{1}{2}\sigma^2 \Sigma_{i,j}\, a_{ij}(x)\, \frac{\partial^2 P}{\partial x_i\, \partial x_j} + \Sigma_i b_i\,(x) \frac{\partial P}{\partial x_i} - rP = \frac{\partial P}{\partial \tau}, \qquad (24)$$

where now we have $P(x, \tau)$, the short-term rate $r \equiv x_1$, the sums run over the n factors, and now the operator $\mathcal{A}$ has been generalized. One also needs similar assumptions about the existence of a stable equilibrium point x^* (an n-vector) of the system $dX_i/d\tau = b_i\,(X)$, $X_i(0) = x_i$, (i=1, 2, ..., n). For example, this equilibrium point assumption holds for both of the CIR Models 1 and 2 . The analogs of the other assumptions (iii), (iv), and (v) in the Introduction are straightforward. Then, the development would proceed along the lines of Section 1.

The small time to maturity expansion proceeds in the same way as indicated in Section 2; one again uses the exponential expansion

$$P(x, \tau) = \exp(-\mathcal{A}\tau)\, P(x,0) = \sum_{n=0}^{N-1} \frac{(-\tau\mathcal{A})^n}{n!} P(x,0) + O(\tau^N).$$

As before, $P(x,0) = 1$ for the default-free discount bond, or one can consider other contingent claims with generalized $P(x,0)$ if the payoff is smooth enough in x. Again, one expects the expansion to be generally asymptotic. Then, it is easy to show that the initial slope of the yield curve is still given by $b_1\,(x)/2$, where the risk-adjusted short-term rate is now described by $dr = b_1\,(x)dt + \sigma\sqrt{a_{11}\,(x)}dz$, and the initial curvature and higher order terms are readily developed. As we indicated earlier in Section 2, this shows in the multi-factor case that the price of the discount bond is still not affected by the drift until $O(\tau^2)$. Hence, under a very general multi-factor setting, we can help justify the recording of short-maturity T-bill yields as an approximate realization of the process $dr = m_1\,(x)dt + \sigma\sqrt{a_{11}\,(x)}dz$, where $m_1\,(x)$ is the drift of the short-term rate.

Finally, the large time to maturity limit is still controlled by the smallest eigenvalue associated with $\mathcal{A}u = \lambda u$ as indicated in Section 3. As before, this eigenvalue satisfies a variational principle which is the generalization of eq. (21) to the n factor dimensions. Note that while the assumption of a common small factor volatility is necessary for the small volatility expansion to be generalized, it is not necessary for either the small or large time to maturity expansions to be applied as indicated.

4.2 Conclusions.

We have provided some new methods for solving the fundamental differential equation that determines the default-free yield curve. In the three limits that we investigate, one can display the dependence of the yields on the parameters of the problem, which is also a benefit that one achieves with exact solutions. However, our methods work when no exact solution is available. In contrast to this, grid-based numerical work makes discovering these dependencies tedious and repetitive.

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Footnotes

1. Interest rate volatility is generally agreed to be time-varying. For example, in the post WWII U.S experience, high interest rate volatility periods have been associated with the above-average rates of 1973-74 or 1980-81. However, a *stationary* interest rate process can still be the generator of this time varying volatility. See, for example the conditional volatility forecasts shown in Fig 1, page 1221 of Chan, et al (1992). At the opposite extreme of the stationary assumption, once one admits truly arbitrary time dependence into the drift, volatility and risk premium functions, the theory almost becomes empty of predictive or explanatory content. An intermediate approach is to incorporate interest rate volatility as a factor in a stationary multi-factor model such as that of Longstaff and Schwartz (1992).

2. Most, but not all, of the published exactly solvable special cases have an equilibrium. As an exception, Dothan's logarithmic (1978) model: $a(r) = r^2$, $b(r) = -Br$ does not have an equilibrium. Note that $p(r)$ is also given by $a^{-1} W^{-1}$, where W is proportional to the *Wronskian* of independent solutions to $\mathcal{A}u = \lambda u$, where $\mathcal{A}$ is the operator on the left-hand-side of eq. (2). The Wronskian plays a fundamental role in the development of eigenfunction expansions for the term structure because its zeros as a function of λ locate the discrete spectrum.

3. After finding the solution to this model, the author discovered that a related multi-factor variation is solved in CIR (1985), eq. (53) page 403. However, the CIR variation still has a short-term rate process with a square root volatility; i.e. $dr = b(r)dt + \sigma\sqrt{r}dz$, a behavior statistically rejected by Chan, et al (1992) over their data period. Independent estimates by the author on this same data set (data graciously supplied by Anthony B. Sanders), using a maximum likelihood estimator, confirm a value for θ near 3/2 over that period. Estimates for θ are fairly insensitive to the precise drift specification. However, in fairness to the "square root" rate process model, I also find a much lower estimate for θ when one extends the data period back to the early 1950's (but post Treasury-Fed accord). Since the behavior of the rate volatility in the model $dr = (Ar - Br^2)dt + \sigma r^{3/2} dz$ has some empirical support, and the model is quite tractable, more empirical testing of eq. (9) would be useful.

4. Because an asymptotic series representation does not determine the function, there may always be an additional piece of the exact solution which vanishes faster than any power of τ as $\tau \to 0$. For example, consider the standard diffusion on $r > 0$, $-\mathcal{A}P = (1/2)(\partial^2 P(r,\tau)/\partial r^2) = \partial P/\partial \tau$, with the reflecting boundary condition $\partial P/\partial r|_{r=0} = 0$. As a simple illustration, take $P(r,0) = \exp(-r)$. Then, the power series expansion of eq. (10) yields the convergent series $P(r,\tau) \approx [1 + (\tau/2)(d^2/dr^2) + (\tau^2/2^3)(d^4/dr^4) + \ldots] e^{-r} = e^{-r+\tau/2}$, where "$\approx$" means "is asymptotically equal to". But, the exact solution is given by $P(r,\tau) = e^{-r+\tau/2} + [e^{r+\tau/2}\,\Phi((-r-\tau)/\sqrt{\tau}) - e^{-r+\tau/2}\,\Phi((-r+\tau)/\sqrt{\tau})]$, where $\Phi(z) = \int_{-\infty}^{z} (2\pi)^{-1/2} \exp(-y^2/2)$. It is easy to see that the additional terms, which are due to the boundary condition, vanish faster than any power of τ for any $r > 0$. Moreover, for small enough τ, the diffusion does not have enough time to "sense" the boundary and so $P(r,\tau) \approx e^{-r+\tau/2}$ will be an accurate numerical approximation.

5. To the author's knowledge, this model was first derived by Merton (1975) as the consequence of his postulated one-sector capital growth model under uncertainty. Merton only solved for the equilibrium interest rate density function. The author has solved the complete dynamical evolution problem that produces the full yield curve (to be submitted).

Table 1.

Yield to Maturity (%) for the Risk-Adjusted Process
$dr = (Ar - Br^2)d\tau + \sigma r^{3/2} dz$
For Various Orders n of the Volatility Expansion.

		Maturity (months)							
σ	Order	12	24	60	120	180	240	360	Infinite
.1	0	3.215	3.426	3.998	4.670	5.056	5.281	5.519	6.000
	1	3.214	3.422	3.970	4.587	4.936	5.143	5.371	5.738
	2	3.214	3.422	3.970	4.586	4.929	5.128	5.339	5.757
	Exact	3.214	3.422	3.970	4.587	4.929	5.127	5.335	5.756
.2	0	3.215	3.426	3.998	4.670	5.056	5.281	5.519	6.000
	1	3.211	3.409	3.890	4.380	4.682	4.888	5.146	4.950
	2	3.211	3.409	3.892	4.368	4.614	4.769	4.976	5.265
	Exact	3.211	3.409	3.892	4.369	4.613	4.751	4.893	5.181

Notes: Time measured in months: $A = .024$, $B = .4$. The short-term rate is 3%.

Table 2.
First Two Terms of the Volatility Expansion for Various Term Structure Models.

Model: $dr = b(r)dt + \sigma\sqrt{a(r)}dz$ References:	$b(r)$	$a(r)$	$x(\tau)$	$P(r,\tau) = P^{(0)}(r,\tau) + \sigma^2 P^{(1)}(r,\tau) + O(\sigma^4)$
Feller (1951), CIR (1985)	$A - Br$	r	$\chi(r,\tau)$	$e^{-r^*\tau} e^{[(r-r^*)(\frac{e^{-B\tau}-1}{B})]} \{1 + \frac{\sigma^2}{2B^2}[r^* a + (r-r^*)b] + O(\sigma^4)\}$
Brennan & Schwartz (1980), Courtadon (1982)	$A - Br$	r^2	$\chi(r,\tau)$	$e^{-r^*\tau} e^{[(r-r^*)(\frac{e^{-B\tau}-1}{B})]} \{1 + \frac{\sigma^2}{2B^2}[(r^*)^2 a + 2r^*(r-r^*)b + (r-r^*)^2 c] + O(\sigma^4)\}$
General case I	$A - Br$	$r^{2\theta}$	$\chi(r,\tau)$	$e^{-r^*\tau} e^{[(r-r^*)(\frac{e^{-B\tau}-1}{B})]} \{1 + \frac{\sigma^2}{2B^2}\int_0^\tau [\chi(r,s)]^{2\theta}(1-e^{-B\cdot(\tau-s)})^2 ds + O(\sigma^4)\}$
Merton (1975)	$Ar - Br^2$	r^2	$r\xi(r,\tau)e^{A\tau}$	$[\xi(r,\tau)]^{1/B} \{1 + \frac{(1+B)\sigma^2}{2A^2} e^{2A\tau} [\xi(r,\tau)]^2 r^2 d + O(\sigma^4)\}$
CIR (1985), this paper.	$Ar - Br^2$	r^3	$r\xi(r,\tau)e^{A\tau}$	$[\xi(r,\tau)]^{1/B} \{1 - \frac{(1+B)\sigma^2}{2B^3}[\ln \xi(r,\tau) + \frac{1}{2}(3-\xi(r,\tau))(1-\xi(r,\tau)] + O(\sigma^4)\}$
General case II	$Ar - Br^2$	$r^{2\theta}$	$r\xi(r,\tau)e^{A\tau}$	$[\xi(r,\tau)]^{1/B} \{1 + \frac{(1+B)\sigma^2}{2A^2} e^{2A\tau} [\xi(r,\tau)]^2 r^{2\theta} \int_0^\tau [e^{As}\xi(r,s)]^{2\theta-2} (1-e^{-A\cdot(\tau-s)})^2 ds\} + O(\sigma^4)\}$

Notes: $P(r,\tau)$ is the price of the pure discount bond maturing in τ periods and r is the short-term interest rate. $x(\tau)$ is the time evolution of the short-term rate in the absence of uncertainty ($\sigma = 0$). In the "General Case" expressions, 2θ can be non-integral. The following expressions in the table are defined here: $r^* = A/B$; $\chi(r,\tau) = r^* + (r-r^*)e^{-B\tau}$; $\xi(r,\tau) = r^*/[r^* + r(e^{A\tau} - 1)]$; $a = \tau - (1/2B)(1-e^{-B\tau})(3-e^{-B\tau})$; $b = -2\tau e^{-B\tau} + (1/B)(1-e^{-2B\tau})$; $c = \tau e^{-2B\tau} + (1/2B)(1-e^{-B\tau})(1-3e^{-B\tau})$; $d = \tau - (1/2A)(1-e^{-A\tau})(3-e^{-A\tau})$

Table 3.

Yield To Maturity (%) for the Risk-Adjusted Process
$dr = (A - Br)d\tau + \sigma r dz$
For Various Orders n of the Time to Maturity Expansion.

	Maturity (months)							
Order	12	24	36	48	60	72	84	96
1	3.216	3.432	3.648	3.864	4.080	4.296	4.512	4.728
2	3.205	3.388	3.549	3.688	3.805	3.899	3.972	4.023
3	3.205	3.391	3.560	3.713	3.854	3.985	4.108	4.226
4		3.391	3.559	3.710	3.847	3.971	4.082	4.181
5			3.559	3.711	3.848	3.973	4.087	4.191
6				3.711	3.848	3.972	4.085	4.187
7						3.973	4.086	4.189
8						3.973	4.086	4.188
final estimate	3.205	3.391	3.559	3.711	3.848	3.973	4.086	4.19

Notes: Time measured in months. $A = (.024)(.03)$, $B = (.4)(.03)$, $\sigma = .1\sqrt{.03}$. The short-term rate is 3%.

Table 4.

Yield To Maturity (%) for the Risk-Adjusted Process
$dr = (A - Br)d\tau + \sigma r dz$
For Various Orders n of the Time to Maturity Expansion.

	Maturity (months)							
Order	12	24	36	48	60	72	84	96
1	3.216	3.432	3.648	3.864	4.080	4.296	4.512	4.728
2	3.203	3.380	3.531	3.657	3.756	3.829	3.877	3.899
3	3.203	3.384	3.544	3.686	3.813	3.929	4.034	4.133
4		3.384	3.543	3.684	3.809	3.919	4.016	4.103
5			3.543	3.684	3.809	3.920	4.019	4.109
6						3.918	4.014	4.097
7						3.919	4.017	4.105
8						3.919	4.017	4.103
final estimate	3.203	3.384	3.543	3.684	3.809	3.919	4.017	4.10

Notes: Time measured in months: $A = (.024)(.03)$, $B = (.4)(.03)$, $\sigma = .2\sqrt{.03}$. The short-term rate is 3%.

Table 5.

Yield To Maturity (%) for the Risk-Adjusted Process

$dr = (Ar - Br^2)d\tau + \sigma r^{3/2} dz$

For Various Orders n of the Time to Maturity Expansion.

	Maturity (months)							
Order	12	24	36	48	60	72	84	96
1	3.216	3.432	3.648	3.864	4.080	4.296	4.512	4.728
2	3.215	3.428	3.640	3.849	4.057	4.263	4.468	4.670
3	3.214	3.422	3.617	3.795	3.951	4.080	4.177	4.236
4	3.214	3.422	3.618	3.797	3.957	4.091	4.197	4.271
5			3.619	3.803	3.974	4.135	4.292	4.456
6			3.619	3.803	3.973	4.132	4.283	4.435
7					3.970	4.119	4.245	4.339
8					3.970	4.12	4.250	4.352
final estimate	3.214	3.422	3.619	3.803	3.970	4.12	4.25	NA
Exact	3.214	3.422	3.619	3.802	3.970	4.123	4.260	4.382

Notes: Time measured in months: $A = .024$, $B = .4$, $\sigma = .1$. The short-term rate is 3%. The exact values are rounded to the digits shown.

Table 6.

Yield To Maturity (%) for the Risk-Adjusted Process

$$dr = (Ar - Br^2)d\tau + \sigma r^{3/2} dz$$

For Various Orders n of the Time to Maturity Expansion

	Maturity (months)							
Order	12	24	36	48	60	72	84	96
1	3.216	3.432	3.648	3.864	4.080	4.296	4.512	4.728
2	3.212	3.417	3.615	3.806	3.989	4.165	4.334	4.496
3	3.211	3.408	3.584	3.731	3.842	3.911	3.930	3.893
4	3.211	3.409	3.587	3.742	3.869	3.967	4.033	4.069
5		3.409	3.590	3.755	3.909	4.066	4.249	4.489
6			3.590	3.752	3.900	4.041	4.185	4.346
7				3.749	3.886	3.987	4.027	3.944
8				3.750	3.889	3.999	4.066	4.056
final estimate	3.211	3.409	3.590	3.75	3.89	4.0	NA	NA
Exact	3.211	3.409	3.589	3.750	3.892	4.016	4.123	4.217

Notes: Time measured in months: $A = .024$, $B = .4$, $\sigma = .2$. The short-term rate is 3%. The exact values are rounded to the digits shown.

Table 7.

Variational Method Estimates for the Yield (%) at Infinite Maturity
for the Risk-adjusted Process $dr = (A - Br)d\tau + \sigma r dz$.

σ (per cent)	0	2	4	6	8	10	12	14	16	18	20	22	24	26	28	30
Yield(α_1 *)	6	5.610	4.972	4.422	3.979	3.620	3.323	3.073	2.859	2.673	2.510	2.365	2.235	2.119	2.013	1.916
Yield(α_1^*, α_2^*)	6	5.610	4.972	4.422	3.977	3.617	3.322	3.067	2.857	2.665	2.500	2.355	2.232	2.107	2.000	1.903

Notes: Time is measured in months. The parameter values are $A = (.024)(.03)$, $B = (.4)(.03)$, and various values for the volatility parameter σ. The first trial function for the lowest eigenvalue was $f(r) = \exp(-\alpha_1 r)$ .The Yield(α_1 *) is the variational bound where α_1 was chosen optimally. The second trial function was $f(r) = (1 + \alpha_2 r^2)\exp(-\alpha_1 r)$, in which case the reported bound is Yield(α_1 *, α_2 *).

Figure Legends

Figure 1.

The figure illustrates the numerical behavior of the small time to maturity expansion in a case where the expansion is known to be asymptotic and divergent. For the risk-adjusted process $dr = (Ar - Br^2)d\tau + \sigma r^{3/2} dz$, $P(r, \tau)$ is given by eq. (13) of the text. With parameter values from Table 2, $\sigma = .2$, and $\tau = 36$ months, the graph shows the value of the partial sums for $P(r, \tau)$ to N terms plotted versus the number of terms N. The expansion imitates a convergent one with $P(r, \tau)$ reaching the value .274668 by $N = 20$ and not changing to six significant figures until N exceeds 110 terms. The value .274668 is the exact one to the digits shown.

Figure 2.

Schematic illustration of the drift function and the risk-adjusted drift function versus the short-term rate. The assumption for the figure is that the market price of interest rate risk is positive. The yield curve will have positive, zero, or negative slope at zero time to maturity when the short-term rate is less than, equal to, or greater than (respectively) the zero of the risk-adjusted drift. The zero of the risk-adjusted drift will be greater than the zero of the raw interest rate process.

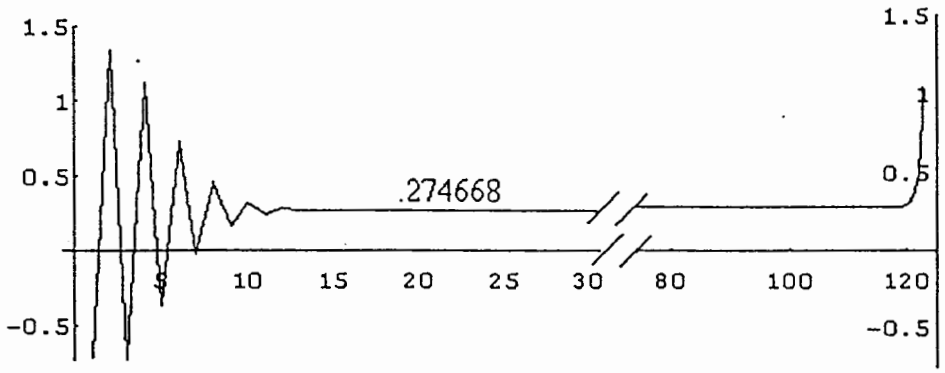
Figure 3.

An illustration of the limiting yield to maturity at infinite maturity for the process $dr = (A - Br)d\tau + \sigma r dz$ versus the volatility scale parameter σ. Parameter values and the data plotted are taken from Table 7. The estimates are made from the variational upper bound eq. (22) with a trial function discussed in the text.

Figure 1.

Pseudo-convergence of the small time
to maturity expansion in the asymptotic case.

The price of the discount bond $P(r, \tau)$



The number of terms in the partial sum eq. (13).

Figure 2.

The Zeros of the Drift and Risk-adjusted Drift.

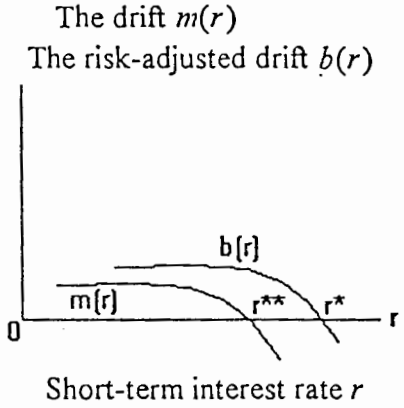


Figure 3.

The Limiting Yield to Maturity for the Process $dr = (A - Br)d\tau + \sigma r dz$.