$$\varphi(\tau) = \frac{2}{\gamma(\tau)}[\exp(\delta\tau) - 1].$$

Then, the exact solution (see Feller[1951]/CIR[1985]) is given by $P(r,\tau) = \psi(\tau)\exp[-\varphi(\tau)r]$. Expanding the exact solution $P(r,\tau)$ in powers of τ and exponentiating gives

$$P(r,\tau) = \exp\left\{-r\tau - (A - Br)\frac{\tau^2}{2} + [(A - Br)B - r\sigma^2]\frac{\tau^3}{6} + O(\tau^4)\right\},$$

which is easily seen to be in agreement with the eq. (11) expansion. In this case, one sees that the expansion is convergent and converges to the exact solution. In *contrast*, consider:

<u>Ex (2.2)</u>

Consider again the equation (9), $\theta = 3/2$ model defined by $a(r) = r^3$, $b(r) = Ar - Br^2$. In the limit $\tau \to 0$, we have the real variable $z(r,\tau) \to +\infty$. Hence, we need to apply the asymptotic formula [see Erdelyi (1953)], valid as $z \to +\infty$, for $N, M = 1,2,3,\ldots,$

$$M(\alpha,\beta,-z) = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)} z^{-\alpha} \sum_{n=0}^{N-1} \frac{(\alpha)_n (\alpha-\beta+1)_n}{n!} z^{-n} + O(|z|^{-\alpha-N})$$

$$+ \frac{\Gamma(\beta)}{\Gamma(\alpha)}(-z)^{\alpha-\beta} e^{-z} \sum_{n=0}^{M-1} \frac{(\beta-\alpha)_n (1-\alpha)_n}{n!}(-z)^{-n} + O(|e^{-z} z^{\alpha-\beta-M}|). \qquad (12)$$

The notation $(\alpha)_n$ is defined by the relations $(\alpha)_0 = 1$, and $(\alpha)_n = \alpha(\alpha+1)\cdots(\alpha+n-1)$, $(n = 1,2,3,\ldots)$. The sums are asymptotic expansions and clearly not convergent since for large n, $(\alpha)_n$ grows like $n!$. The first summation for $M(\alpha,\beta,-z)$, and using eq. (9), implies that

$$P(r,\tau) = \sum_{n=0}^{N-1} \frac{(\alpha)_n(\alpha-\beta+1)_n}{n!} z(r,\tau)^{-n} + O(|z(r,\tau)|^{-N}). \qquad (13)$$

Expanding $z(r,\tau)^{-1} = (\sigma^2 r/2)(\tau + A\tau^2/2 + A^2\tau^3/6) + O(\tau^4)$ and substituting into the equation directly above yields

$$P(r,\tau) = \exp\left\{-r\tau - (Ar - Br^2)\frac{\tau^2}{2} + [-A^2 r + 3ABr^2 - 2B^2 r^3 + \sigma^2(1+B)r^3]\frac{\tau^3}{6} + O(\tau^4)\right\}$$