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Volatility Jump-diffusions[1]

Many readers know my book "Option Valuation under Stochastic Volatility". In that, I explored stochastic volatility in some depth, but limited myself to processes with continuous sample paths. Currently, I am working on a follow-up Volume II, covering jump processes. Indeed, in many of my recent columns, I have discussed problems where the security price can jump.

But, if the security price can jump, shouldn't we expect the (diffusion) volatility to sometimes jump as well? This is a natural generalization and has been considered by a number of researchers. Some notable studies combining theory and empirical support for the concept are Duffie, Pan, and Singleton [76], Eraker, Johannes, and Polson [84], and Bakshi and Cao [13].

In this month's column, I explore a key issue in jumping volatility. Specifically, I consider a very basic mathematical problem – namely, if volatility follows a jump-diffusion process, then how do you compute its stationary (long-run) probability density? I provide both a general approach and two specific examples. One example can be solved in closed form, but the more important one cannot. I call the second one 'more important' because my guess (more on that below) is that it is likely to be a better fit to real-world option prices. I call my preferred model the *GARCH jump-diffusion*. Once you see how to handle that one (numerically), you will have a toolkit for many jump-diffusions.

While I use the language of volatility, the problem I tackle has wider applicability in finance. In stochastic theories of finance, consider the difference between (traded) security prices and (non-traded) parameters. Equity prices, for example, usually are modeled by stochastic processes with exponential growth plus noise – they don't have a stationary density. Bond prices don't grow exponentially, yet they don't have a stationary density either because the bonds mature. But *parameters*, such as volatility, or interest rates, are completely different. These parameters almost always have a stationary density, at least in models, because we expect them to be recurrent (i.e., mean-reverting). So, this work applies generally to mean-reverting stochastic parameters that follow jump-diffusion processes.

22.1 A class of stock price models

Consider a class of continuous-time *jump-diffusion* models for stock prices which admit both stochastic volatility and jumps. We suppose jump events affect both the stock price and volatility simultaneously.

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The jumps are driven by a compound Poisson process, part of a larger class of "marked point processes". Using that language, the two jump amplitudes (ξ_1, ξ_2) (one for the log stock price and one for the volatility) are the "marks". When a jump event occurs, the jump amplitudes (ξ_1, ξ_2) are drawn from some joint, ξ-dependent, density $m(V_t, \xi_1, \xi_2)$.

We have an underlying probability measure representing the real-world or some risk-adjusted one. Based upon that, we consider processes described by the SDE:

$$d(\log S_t) = \mu_t \, dt + \sigma_t dW_t + \xi_1 dN_t$$
$$dV_t = b(V_t)dt + a(V_t)dB_t + \xi_2 \, dN_t.$$

Here $V_t = \sigma_t^2$ is the diffusive volatility, and (W_t, B_t) are Brownian motions, possibly correlated.

The symbol dN_t is a jump counting process equal to 1 when there is a jump event (and otherwise zero). The expected number of jumps, in a small interval looking forward from time t, is given by $\mathrm{E}\,[dN_t \,|\mathcal{F}_t\,] = \lambda(V_t)dt$, where $\mathcal{F}_t$ is the filtration or information flow, and λ_t is a Poisson intensity. In our case, $\mathcal{F}_t$ simply represents the continuous observation of the stock price up until time t. (From that, the instantaneous diffusion volatility is also known for any time up until t).

The corresponding marginal jump distribution for the volatility alone has a density $m(V_t, \xi_2)$. Since volatility is positive, a general restriction is that volatility jumps must preserve this.

22.2 Adjoint operator

Consider the volatility (variance rate) parameter V_t. Suppose it follows a 1-D jump-diffusion process on the positive real axis (under the real-world probability measure). Let the transition density for this process be $p(t_0, T; V_0, V)$. If the volatility process is time-homogeneous (admittedly a strong assumption), then a sensible model for this will admit a stationary transition density $\Psi(V) = \lim_{T\to\infty} p(t_0, T; V_0, V)$, independent of the starting value V_0. Otherwise, the volatility would typically tend to zero or infinity – contrary to most notions about volatility. The same could be said for many stochastic "parameters", and other financial processes, such as interest rates r_t. Even without time-homogeneity, parameters are usually modeled as *mean-reverting* or *recurrent* for a variety of good reasons. In contrast, mean-reversion is *not* a property we would expect for many security prices, like stock prices, because we usually expect long-run exponential growth there. So, our discussion here applies generally to any mean-reverting, stationary, stochastic parameter.

Consider the 1-D diffusion case, where a process X_t follows $dX = b(X)dt + a(X)dB(t)$ on some interval (x_0, x_1) of the real axis. (I will use X_t and V_t interchangeably). It is well-known in the diffusion case (see Vol. I, Chapter 9) that a stationary transition density, if it exists, is an integrable solution to:

$$\boldsymbol{\mathcal{A}}^{\dagger}\Psi(x) := \frac{1}{2}\frac{d}{dx^2}[a^2(x)\Psi(x)] - \frac{d}{dx}[b(x)\Psi(x)] = 0.$$

We call $\boldsymbol{\mathcal{A}}^{\dagger}$ the adjoint generator. In addition, if a boundary point x_b can be reached by the process (starting from the interior) in finite time, then we need to reflect it at the boundary to allow for stationary behavior. More generally, we suppose the *zero-flux* boundary condition holds at each boundary, reachable or not:

$$\left.\frac{1}{2}\frac{d}{dx}[a^2(x)\Psi(x)] - [b(x)\Psi(x)]\right|_{x_b} = 0,$$

where x_b is one or both of (x_0, x_1).

Our goal in this section is to derive the analogous operator $\boldsymbol{\mathcal{A}}^\dagger$ for a stationary jump-diffusion process:

$$dX_t = b(X_t^-)dt + a(X_t^-)dW(t) + \gamma(X_t^-,\xi)\,dN(d\xi,dt).$$

We suppose a positive processes $X(t) \geq 0$ because that is the typical application. The process is determined by its characteristics (or coefficients), (b,a,γ,λ,m), where m is the jump size density, *and* the specification of what happens should the process reach or cross a boundary. To avoid boundary complications as far as possible, we restrict the class of processes considered in examples to those whose characteristics make both $X = 0$ and $X = \infty$ unreachable in finite time, starting from the interior $X(0) = x > 0$. With that, the corresponding generator is:

$$\begin{aligned}\boldsymbol{\mathcal{A}}\,f &= \frac{1}{2}a^2(x)\frac{\partial^2 f}{\partial x^2} + b(x)\frac{\partial f}{\partial x}\\ &\quad + \lambda(x)\int_0^\infty [f(x+\gamma(x,\xi)) - f(x)]\,m(x,\xi)\,d\xi. \qquad (22.1)\end{aligned}$$

Since the process is *stationary*, the transition density can be reduced to one time argument: $p(t,T;x,y) = \tilde{p}(T-t;x,y) = \tilde{p}(\tau;x,y)$, using $\tau = T-t$. In that case, the transition density is a solution to

$$\frac{\partial}{\partial\tau}\,\tilde{p}(\tau;x,y) = \boldsymbol{\mathcal{A}}_x\,\tilde{p}(\tau;x,y). \qquad (22.2)$$

(The subscript on the operator indicates that it acts upon x, not y). As in the pure diffusion case, the stationary density has the following invariance property:

$$\Psi(y) = \int_0^\infty \Psi(x)\,\tilde{p}(\tau;x,y)\,dx.$$

Taking the τ derivative of both sides, and applying (22.2), we have

$$0 = \int_0^\infty \Psi(x)\,\boldsymbol{\mathcal{A}}_x\tilde{p}(\tau;x,y)\,dx.$$

Now the goal should be familiar from the diffusion case: one wants to do parts integrations to obtain a new relation in the form:

$$0 = \text{Boundary terms} + \int_0^\infty [\boldsymbol{\mathcal{A}}^\dagger\Psi(x)]\,\tilde{p}(\tau;x,y)\,dx.$$

(We drop the subscript on $\boldsymbol{\mathcal{A}}^\dagger$ when, as in the last expression, there is no ambiguity). If the boundary terms vanish (which they do, typically), then we observe that the term $\tilde{p}(\tau;x,y)$ is positive and can be made to concentrate near any point x_0 by choosing $y = x_0$ and letting $\tau \to 0$. Since x_0 is arbitrary, this means that the only way the integral can vanish is that

$$\boldsymbol{\mathcal{A}}^\dagger\Psi(x) = 0,$$

which gives us an equation to solve for the stationary density. Since these manipulations are given in Vol. I (Chapter 9) for the diffusion case, I will concentrate on just the new jump term that appears. To keep things simple, I will take the case where $\gamma(x,\xi) = \xi$, and $m(x,\xi) = m(\xi)$, with support on $\xi \geq 0$ only. In that case, the jump-diffusion generator has the form

$$\begin{aligned}\boldsymbol{\mathcal{A}}\,f &= \boldsymbol{\mathcal{A}}_c f + \boldsymbol{\mathcal{B}}\,f\\ &= \frac{1}{2}a^2(x)\frac{\partial^2 f}{\partial x^2} + b(x)\frac{\partial f}{\partial x} + \lambda(x)\int_0^\infty [f(x+\xi) - f(x)]\,m(\xi)\,d\xi,\end{aligned}$$

where $\boldsymbol{\mathcal{A}}_c$ is the familiar diffusion generator (first two terms) and $\boldsymbol{\mathcal{B}}$ is the jump-term generator.

Now consider for *any* smooth test function $f(x)$, the expression

$$\int_0^\infty \Psi(x)\,(\boldsymbol{\mathcal{B}}\,f)(x)\,dx = \int_0^\infty \int_0^\infty \Psi(x)\lambda(x)\,[f(x+\xi)-f(x)]\,m(\xi)\,d\xi\,dx$$
$$= \int_0^\infty \int_x^\infty \Psi(x)\lambda(x)f(\xi')m(\xi'-x)\,d\xi'\,dx - \int_0^\infty \Psi(x)\lambda(x)f(x)\,dx$$
$$= \int_0^\infty \left[\int_0^x \lambda(\xi)m(x-\xi)\Psi(\xi)\,d\xi - \lambda(x)\Psi(x)\right] f(x)\,dx.$$

Notice that to get from the 2nd line to the third, we have taken $x \to 0$ in the integration limit, which makes no difference since $m(\xi)$ vanishes for $\xi < 0$. Then, we have relabeled $\xi' \to x$ and $x \to \xi$, and exchanged integration orders. These rearrangement apply to our problem by taking $f(x) \to \tilde{p}(\tau; x, y)$, where the other parameters (τ, y) play no role in the manipulations. Finally, recalling from Vol. I (Chapter 9) the diffusion and the parts boundary terms, we have our first result:

Proposition 22.1 (stationary density) *Suppose X_t is a jump-diffusion on the positive axis, with non-negative marks $\xi \sim m(\xi)$, jump intensity $\lambda(X_t)$ and the SDE:*

$$dX_t = b(X_t)dt + a(X_t)dB_t + \xi dN_t.$$

Suppose there are no boundary fluxes. Then, if a stationary density $\Psi(x)$ exists for this process, it satisfies the equation $\boldsymbol{\mathcal{A}}^\dagger\Psi(x) = 0$, where

$$\boldsymbol{\mathcal{A}}^\dagger\Psi(x) = \frac{1}{2}\frac{d^2}{dx^2}[a^2(x)\Psi(x)] - \frac{d}{dx}[b(x)\Psi(x)] + \int_0^x \lambda(\xi)m(x-\xi)\Psi(\xi)\,d\xi - \lambda(x)\Psi(x), \qquad (22.3)$$

and the zero-flux boundary conditions:

$$\lim_{R\to\infty}\lim_{\epsilon\to 0}\frac{1}{2}\frac{d}{dx}[(a^2(x)\Psi(x)] - b(x)\Psi(x)\Big|_\epsilon^R = 0. \qquad (22.4)$$

22.3 Parameterized class of models

In Vol. I, we considered the parameterized class of (volatility) diffusions:

$$dV_t = (\omega - \theta V_t)dt + \sigma_V V_t^\phi dB_t,$$

where $(\omega, \theta, \sigma_V, \phi)$ are constant parameters. In particular, we studied in some detail the particular cases: $\phi = (1/2, 1, 3/2)$, which we will call here the "square-root model", "GARCH diffusion", and the "linear-drift 3/2-model" respectively. We take $\theta > 0$ and note that $2\phi = n = (1, 2, 3)$, an integer. Additional parameter assumptions are given at the end of Sec. 22.4.

Which value for ϕ is the 'best'? At this writing, and based only on what I have seen, empirical work using volatility jumps (in continuous-time models) exclusively try to fit the square-root model. (When there are no jumps, this model is also known as Heston's model). The reason for that is because the square root model, in many guises, is solvable.

Yet, when there are no jumps, if the parameter ϕ is left to float in empirical fits to prices series such as the S&P500 index, researchers typically find estimates in the range $1 < \phi < 3/2$. As a model of stochastic volatility, the square-root model is frequently rejected in empirical studies where it is placed in competition against larger values of ϕ.

However, the inclusion of jumps in both price changes and volatility would be expected to lower these empirical estimates of ϕ. Based upon the evidence without the jumps, I suspect that, of our three choices above, $\phi = 1$ would prove to be the best fit for the well-studied SPX series.[2]

Moreover, $\phi = 1$, in the absence of jumps, is a diffusion limit of an appealing discrete-time GARCH model. In addition, you will see below why having $2\phi =$ an integer is very convenient. Finally, the cases $\phi > 1$ are theoretically troublesome for various reasons.

Given all that, you can now see why my intuition is that the case $\phi = 1$ is the 'more important' one.

22.4 Stationary transition density

In this section, we will *set-up* the problem of finding the stationary density $\Psi(V)$ for the jump-diffusion extension of each of these three cases, with constant jump intensity λ. However, we will explicitly carry it through only for the first two. We now have

$$dV_t = (\omega - \theta V_t)dt + \sigma_V V_t^{\phi} dB_t + \xi dN_t. \tag{22.5}$$

With $\omega > 0$ and $\phi > \frac{1}{2}$, the boundary $X = 0$ is unreachable by Table 2.6 of Ch. 2. With $\omega > 0$ and $\phi = \frac{1}{2}$, the absence of a boundary flux at $X = 0$ for our solution will be confirmed separately. Similarly, it can be shown that $X = \infty$ is unreachable. Hence, there are no boundary fluxes and Proposition 22.1 applies. This means we seek a normalizable solution, on $V \geq 0$ to:

$$0 = \frac{1}{2}\sigma_V^2 \frac{d^2}{dV^2}[V^n \Psi(V)] - \frac{d}{dV}[(\omega - \theta V)\Psi(V)] + \lambda \left[\int_0^V m(V-\xi)\Psi(\xi)\, d\xi - \Psi(V)\right].$$

Removing the dependence upon σ_V. By multiplying the above equation by $2/\sigma_V^2$ and defining new parameters: $(\tilde{\theta}, \tilde{\omega}, \tilde{\lambda})$, we can eliminate σ_V from the equation. Here $\tilde{\theta} = 2\theta/\sigma_V^2$, etc. However, to avoid needless clutter in notation, I am going to do this without writing the "tildes" on the new parameters. The reader can easily restore them, or instead, think of the solutions given as requiring that $\sigma_V = \sqrt{2}$.

A transform approach. Since the last term is a convolution, this suggests a Laplace transform approach, defining $F(s) = \mathcal{L}[\Psi](s) = \int_0^\infty \mathrm{e}^{-sV}\Psi(V)dV$, and $\hat{m}(s) = \mathcal{L}[m](s)$. Notice that we have, after parts integrations,

$$\int_0^\infty \mathrm{e}^{-sV}\left\{\frac{d^2}{dV^2}[V^n\Psi(V)] - \frac{d}{dV}[(\omega - \theta V)\Psi(V)]\right\} dV = G_0 - s\,\omega F(s) - s\,\theta F'(s) + s^2 \int_0^\infty \mathrm{e}^{-sV} V^n \Psi(V)\, dV. \tag{22.6}$$

The constant G_0, the boundary flux at the origin, is given by

$$G_0 = \lim_{V\to 0}\left\{-\frac{d}{dV}[V^n\Psi(V)] + (\omega - \theta V)\Psi(V)\right\}.$$

[2] Note added: since publication of this article in 2004, VIX options have become very active and a good tool for resolving SPX volatility models. See Ch. 4.

But G_0 is simply our boundary condition (22.4) again and will vanish.[3] Also, we can write:

$$\int_0^\infty e^{-sV} V^n \Psi(V)\, dV = (-1)^n \frac{d^n}{ds^n} F(s). \tag{22.7}$$

With the jump term added, and accepting that $G_0 = 0$ for the moment, this means that the Laplace transform of the stationary density must solve the ODE:

$$(-1)^n s^2 \frac{d^n}{ds^n} F(s) - s\,\theta F'(s) - [s\,\omega - \lambda(\hat{m}(s) - 1)]\, F(s) = 0, \qquad (n = 1, 2, 3). \tag{22.8}$$

Remark on pseudo-differential operators. We have defined equation (22.8) under the assumption that $n = 2\phi$ is an integer. It may also makes sense for n an arbitrary real number. In that case, you have a derivative operator raised to some real power. What is the meaning of that? It is well-defined through the Laplace transform of equation (22.7) – and the operator itself is called a "Pseudo-differential operator (PDO)." PDOs are simply linear operators that act as multipliers through the Laplace or Fourier transform. Notice that all our jump-integrals are PDOs; they turn into function multipliers in "transform space".

Boundary conditions. What are the boundary conditions for equation (22.8)? For each case, we have $F(0) = 1$. For $n = 1$, equation (22.8) is a first order equation and no further conditions are needed. For $n = 2$, we have a second order equation, so we need more information. But, we know that $F(s)$ must tend to zero as $s \to \infty$ since $\Psi(V)$ is a probability density. For $n = 3$, we have a third order equation, and potentially we need to know even more.

The long-run volatility. Consider the expected future volatility of our process $\bar{V}_t := \mathrm{E}\,[V_t|V_0] = f(V_0, t)$. We will restrict ourselves to the class of processes where this is finite for all t including $t = \infty$. Then, we define the long-run volatility, by $\bar{V}_\infty = \lim_{t\to\infty} \bar{V}_t$. But, since our process admits a stationary density (V_t is *ergodic*), we can also obtain the long-run volatility from $\bar{V}_\infty = \int_0^\infty V\Psi(V)dV$. The interplay between these two formulations will be important to our computations.

Now $f(x,t)$ can be found by solving the integro-PDE problem

$$\frac{\partial f}{\partial t} = \frac{1}{2} x^{2\phi} \frac{\partial^2 f}{\partial x^2} + (\omega - \theta x) \frac{\partial f}{\partial x} + \lambda \int_0^\infty [f(x+\xi, t) - f(x,t)]\, m(\xi)\, d\xi, \tag{22.9}$$

with the initial condition $f(x, 0) = x$.

However, the solutions to such problems are not always unique. Keeping that in mind, consider the solution *ansatz* $f(x,t) = b + \exp(-at)(x - b)$, where a and b are constants to be shortly determined. First of all, this proposed solution clearly satisfies the initial condition. Next, substitution into (22.9) shows that it is indeed a solution if $-a(x - b) = \omega - \theta x + \lambda \bar{\xi}$, where $\bar{\xi} = \int_0^\infty \xi\, m(\xi)\, d\xi$. In other words, since this must hold for all x, we have $a = \theta$ and $b = (\omega + \lambda \bar{\xi})/\theta$. If this solution were unique, then clearly $\bar{V}_\infty = b$. As it turns out, the solution is *not* unique for $\phi > 1$ and this is not the solution. Moreover, for $\phi < 1/2$, we have the problem that the process can reach $X = 0$ in finite time and need a boundary condition there. This is easy to establish in the diffusion case, and the additional jumps shouldn't change that. But, to summarize, we have the useful result:

$$\bar{V}_\infty = -F'(0) = \int_0^\infty V\Psi(V)\, dV = \frac{1}{\theta}(\omega + \lambda\bar{\xi}), \quad \frac{1}{2} \le \phi \le 1. \tag{22.10}$$

This last relation suggests, and we shall, adopt the following assumptions on parameters:

Model Assumptions. For each ϕ, we assume that: $\omega > 0$, $\theta > 0$, and $\lambda \ge 0$. If $\lambda > 0$, then $0 < \bar{\xi} < \infty$.

[3] We are able to separately verify this for the square-root model.

22.5 The square-root jump-diffusion model

In this case we have $n = 2\phi = 1$ and (22.8) reads

$$(s^2 + \theta s)F'(s) + [\omega s - \lambda(\hat{m}(s) - 1)]\, F(s) = 0. \tag{22.11}$$

With the initial condition $F(0) = 1$, this equation is easily integrated and the solution is

$$F(s) = \exp\left\{ - \int_0^s \frac{\omega u - \lambda[\hat{m}(u) - 1]}{u^2 + \theta u}\, du \right\}. \tag{22.12}$$

This expression is well-defined since (i) analyticity of $\hat{m}(u)$ at $u = 0$ ensures integrability there, and (ii) $\theta > 0$ ensures that the denominator in the integrand will never vanishes for $u > 0$.

Example: exponentially distributed jumps. We take $m(\xi) = \eta\, \mathrm{e}^{-\eta\xi}$, for $\xi \geq 0$, so that $\hat{m}(s) = \eta/(s + \eta)$. A straightforward integration then yields

$$F(s) = \left[\frac{s+\theta}{\theta}\right]^{-\omega - \frac{\lambda}{\eta-\theta}} \left[\frac{s+\eta}{\eta}\right]^{\frac{\lambda}{\eta-\theta}}$$

Now the Laplace inversion may be accomplished with a formula from Vol. I (Ch.11, (2.8)), which we recall here for the reader's convenience:

$$\mathcal{L}^{-1}[s^{-a}(s-c)^{-b}](t) = \frac{t^{a+b-1}}{\Gamma(a+b)} M(b, a+b, ct), \qquad \Re(a+b) > 0.$$

Here $M(a, b, z)$ is a confluent hypergeometric function. Using that, we obtain the stationary density:

$$\Psi(x) = C x^{\omega-1} \mathrm{e}^{-\theta x} M[b, \omega, (\theta - \eta)x], \tag{22.13}$$

$$\text{where} \quad b = -\tfrac{\lambda}{\eta-\theta}, \quad \text{and} \quad C = \tfrac{\theta^\omega}{\Gamma(\omega)} \left(\tfrac{\theta}{\eta}\right)^{\lambda/(\eta-\theta)}.$$

Let's do some checks on this solution. First, we want to check the normalization; i.e., does $\int \Psi(x) dx = 1$?. The check can be done with the formula[4]

$$\int_0^\infty \mathrm{e}^{-st} t^{b-1} M(a, c, kt)\, dt = \frac{\Gamma(b)}{s^b} \times {}_2F_1\left(a, b; c; \frac{k}{s}\right),$$

$$|s| > |k|, \quad \Re b > 0, \quad \Re s > \max(0, \Re k).$$

Appearing above is ${}_2F_1(a, b; c; z)$, the Gauss hypergeometric function. Using Pochhammer's symbols $(a)_0 = 1, (a)_1 = a, (a)_2 = a(a+1)$, etc., this function is defined near the origin by the series

$${}_2F_1(a, b; c; z) = \sum_{n=0}^\infty \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}, \quad |z| < 1.$$

Applying the formula, we obtain

$$\int_0^\infty \Psi(x)\, dx = \left(\tfrac{\eta}{\theta}\right)^{-\lambda/(\eta-\theta)}\ {}_2F_1\left(-\tfrac{\lambda}{\eta-\theta}, b; b; 1 - \tfrac{\eta}{\theta}\right) = 1$$

because ${}_2F_1(a, b; b; z) = (1-z)^{-a}$.

Next, recall that we proceeded on the assumption that the boundary flux G_0 at the origin vanished, where we now have

$$G_0 = \lim_{x \to 0} \left\{ -\frac{d}{dx}[x\Psi(x)] + (\omega - \theta x)\Psi(x) \right\}.$$

[4] Section 7.621 in [115].

Now for small x, we have $\Psi(x) \sim c_1 x^{\omega-1} + c_2 x^{\omega}$, where $\omega > 0$. So $G_0 = 0$, as previously asserted, but notice that the limit is not trivial, since it involves the cancellation of terms. In other words, the flux expression must be taken *as a whole*, since the individual parts might not vanish separately.

Finally, let's check the limit where there are no jumps. In that case, $M[0, b, z] = 1$. So we find $\Psi(x) = Cx^{\omega-1}\mathrm{e}^{-\theta x}$, which is correct and easily checked by integrating the Kolmogorov forward equation.

22.6 The GARCH jump-diffusion model

In this case we have $n = 2\phi = 2$ and (22.8) reads

$$F''(s) - \frac{\theta}{s}F'(s) - \left[\frac{\omega}{s} - \lambda\frac{(\hat{m}(s) - 1)}{s^2}\right] F(s) = 0. \tag{22.14}$$

In general, when there are jumps, equation (22.14) is not going to lead to an ODE for a recognizable special function. (Yet, see footnote 5.) However, armed with a powerful symbolic algebra system, such as Mathematica, we can numerically solve the ODE and do the Laplace inversion. Our algorithm will rely upon the existence of series and/or asymptotic solutions near singular points, combined with the ability to solve the ODE numerically. (Mathematica's `NDSolve` routine). Our approach will enable us to develop solutions in the complex s-plane, which we need because of the Laplace inversion. Our general approach should handle many cases of interest in finance.

No jumps. An important special case for our method is when there are no jumps. If $\lambda = 0$, it is straightforward to show that the one solution to (22.14) that (i) is bounded at $s = 0$ and (ii) decays as $s \to +\infty$ along the real axis is

$$F(s) = \frac{2}{\Gamma(\nu)}(\omega s)^{\nu/2} K_\nu(2\sqrt{\omega s}), \quad \text{using} \quad \nu = 1 + \theta, \tag{22.15}$$

where $K_\nu(z)$ is a modified Bessel function. The Laplace inversion is known, and given by

$$\Psi(x) = \mathcal{L}^{-1}\left[\frac{2}{\Gamma(\nu)}(\omega s)^{\nu/2} K_\nu\left(2\sqrt{\omega s}\right)\right](x) = \frac{\omega^\nu}{\Gamma(\nu)} x^{-\nu-1}\mathrm{e}^{-\omega/x}, \tag{22.16}$$

which is correct and again easily verified from the Kolmogorov forward equation.

Exponential jumps. For the remainder of this section, we assume that $\lambda > 0$. In particular, to focus the ideas, we consider again the exponential jump distribution. This is merely for illustration as you will see that our approach does not rely upon this specific functional form. With that illustration, (22.14) becomes:[5]

$$F''(s) - \frac{\theta}{s}F'(s) - \left[\frac{\omega}{s} + \frac{\lambda}{s(s+\eta)}\right] F(s) = 0. \tag{22.17}$$

Solutions near regular and irregular singular points. We have written (22.17) in the standard form

[5] The ODE (22.17) has been named – it's called a *confluent Heun equation.* This does not help us directly as this special function is not built-in to Mathematica, our computing environment for this problem. Nevertheless, I would like to thank Prof. Reinhard Schäfke of the Univ. of Strasbourg for suggesting that my ODE was a confluent Heun equation.

$$F''(s) + P(s)F'(s) + Q(s)F(s) = 0. \tag{22.18}$$

Let's recall briefly how one constructs series solutions to such equations near any arbitrary point s_0. If $P(s)$ and $Q(s)$ are regular[6] at $s = s_0$, then s_0 is called an *ordinary point* of the differential equation. Near an ordinary point, the general solution of the ODE is an ordinary Taylor series $F(s) = \sum_{n \geq 0} c_n (s - s_0)^n$, with a radius of convergence out to the nearest singularity.

If s_0 is not a regular point, but $(s - s_0)P(s)$ and $(s - s_0)^2 Q(s)$ are regular, then s_0 is called a *regular singular point.* For (22.17), both $s = 0$ and $s = -\eta$ are regular singular points.

Near a regular singular point, one can construct a series solution by the Method of Frobenius. The method guarantees that there is always at least one solution of the form $F_1(s) = (s - s_0)^{\nu_1} \sum_{j \geq 0} c_j (s - s_0)^j$, where ν_1 is not necessarily an integer and $c_0 \neq 0$. By substituting this form into the ODE, one obtains a quadratic equation, called the *indicial* equation for ν, whose solutions we label (ν_1, ν_2), where $\nu_1 \geq \nu_2$. If $\nu_1 - \nu_2 \neq 0$ or a positive integer, then there is a second linearly independent solution $F_2(s)$ of the same form that uses ν_2. However, if $\nu_1 - \nu_2 = N$, then the second linearly independent solution is of the form $F(s) = (s - s_0)^{\nu_2} \sum_{j \geq 0} c_j (s - s_0)^j + C F_1(s) \log |s - s_0|$. If $N > 0$, then C may be zero or it may not be, but if $N = 0$ (i.e., $\nu_1 = \nu_2$), then $C \neq 0$.

For (22.17), the indicial equation is $\nu(\nu - 1 - \theta) = 0$. With our maintained assumption that $\theta > 0$, we have $\nu_1 = 1 + \theta$ and $\nu_2 = 0$. For θ to be an integer would be exceptional, so for simplicity we assume that it is not. To simplify the notation, from this point, we shall simply write ν for $1 + \theta$ and drop the subscript '1'. So we can proceed knowing that ν is not an integer.

Then, one linearly independent solution to (22.17) has the form $F_1(s) = s^\nu \sum_{j \geq 0} a_j s^j$ and a second has the form $F_2(s) = \sum_{j \geq 0} b_j s^j$. Using $\beta = \lambda + \omega\eta$, we find the three-term recurrence relations:

$$\eta\, a_j [j(j + \nu)] + a_{j-1}[(j - 1)(j + \nu - 1) - \beta] - \omega a_{j-2} = 0, \tag{22.19}$$

$$\eta\, b_j [j(j - \nu)] + b_{j-1}[(j - 1)(j - \nu - 1) - \beta] - \omega b_{j-2} = 0. \tag{22.20}$$

These relations hold for any j with the convention that $a_{-1} = a_{-2} = \cdots = 0$ and similarly for b_{-j}.

Now $F_1(s) \sim a_0 s^{1+\theta}$ as $s \to 0$, while $F_2(s) \sim b_0 + b_1 s + O(s^2)$. To obtain a particular solution with $F(0) = 1$, we choose $b_0 = 1$, $a_0 = 1$, and then our solution must have the form $F(s) = F_2(s) + A F_1(s)$, where A is an undetermined constant at this point. All the higher order coefficients, a_n and b_n are determined by the recurrence relations. Note in particular that $b_1 = -(\omega + \lambda/\eta)/\theta$, and so $F'(0) = b_1$, which is our expected result based upon (22.10). This solution for $F(s)$, with one undetermined constant A, will converge everywhere in the complex s-plane within the circle $|s| < \eta$, since the nearest singularity to the origin lies at $s = -\eta$.

The point at infinity. The point at infinity often presents special problems. In general, one can classify it in the same way, by taking $x = 1/s$ and then considering the behavior of the transformed ODE near $x = 0$. If one does this for (22.17), for example, writing $F(s) = G(x)$, then one finds the ODE:

$$G''(x) - \frac{2 + \theta}{x} G'(x) - \left[\frac{\omega}{x^3} + \frac{\lambda}{x^2(1 + \eta x)} \right] G(x) = 0. \tag{22.21}$$

This shows us that $\omega > 0$ prevents the transformed equation from having a regular singular point at $x = 0$. Since $x = 0$ corresponds to $s = \infty$, we say that $s = \infty$ is an *irregular* singular point. How does one proceed?

[6] analytic and single-valued.

As we discussed in Vol. I, Chapter 10, if the irregular singular point is not an essential singularity, but is of *finite rank*, defined below, then significant progress can still be made. Near an irregular singularity of finite rank, there is no longer a convergent series solution. Nevertheless, it was discovered by Thomé that near such singularities there are still formal (asymptotic) solutions to the ODE. These asymptotic solutions, even though given by divergent sums, are in fact sufficient to solve our problem, as we shall see.

To specify what we mean by the rank of an irregular singularity, we need to re-write (22.18) again by eliminating $P(s)$. This can always be done by introducing the new function $u(s)$ through $F(s) = u(s) \exp\{-(1/2) \int P(s) ds\}$. Then, $u(s)$ satisfies

$$u''(s) - V(s)u(s) = 0,$$

$$\text{where} \quad V(s) = -Q(s) + \frac{1}{2}P'(s) + \frac{1}{4}P^2(s).$$

This is Liouville standard form, and differs only by a constant from the Schrödinger equation for the physicists. Then, a sufficient condition for $s = \infty$ to be an ordinary point is $V(s) = O(s^{-4})$ as $s \to \infty$. A sufficient condition for $s = \infty$ to be a regular singular point is $V(s) = O(s^{-2})$ as $s \to \infty$. If $V(s)$ fails both of these tests, but there is a least *integer* r, such that $V(s) = O(s^{2r-2})$ as $s \to \infty$, then r is the rank of the irregular singularity. For the exponential jump model, $P(s) = -\theta/s$, so that $F(s) = s^{\theta/2}u(s)$, and the Liouville standard form has the 'Schrödinger potential'

$$V(s) = \frac{\omega}{s} + \frac{\lambda}{s(s+\eta)} + \frac{\theta(\theta+2)}{4s^2} = O(s^{-1}) \quad \text{as} \quad s \to \infty.$$

The singularity test yields $r = 1$; the point at infinity is an irregular singular point of rank one.

Using the method of Thomé, we find two asymptotic solutions. One of them diverges exponentially, and so cannot be the solution we seek. The solution we seek will vanish as s grows large, and indeed the other Thomé solution has the desired behavior:

$$F(s) = B s^{\nu/2-1/4} e^{-2\sqrt{\omega s}} \left\{ 1 + \frac{c_1}{\sqrt{\omega s}} + \frac{c_1 c_2}{2\omega s} + O(s^{-3/2}) \right\}, \quad \text{as} \quad s \to +\infty,$$

$$\text{using} \quad c_1 = \lambda + \frac{3}{16} + \frac{1}{4}(\nu^2 - 1), \quad \text{and} \quad c_2 = \lambda - \frac{5}{16} + \frac{1}{4}(\nu^2 - 1).$$

We again have an undetermined constant B.

At this point, we have the following *connection problem*: determine the constant A in the solution near the origin, so that solution, when analytically continued outside the circle $|s| < \eta$, becomes the 'small' asymptotic solution we have just exhibited.[7] To solve that problem, we proceed as follows.

Consider the function $G(s) = F'(s)/F(s)$. That function satisfies a first order, nonlinear differential equation, the *Riccati equation*:

$$G'(s) + [G(s)]^2 - \frac{\theta}{s} G(s) - \left[\frac{\omega}{s} + \frac{\lambda}{s(s+\eta)} \right] = 0. \tag{22.22}$$

So, we pick two positive real values $s = s_1, s_2$, where $0 < s_1 < \eta$ and $s_2 >> 1$. The idea is that near s_2 our asymptotic solution will apply, which implies that

[7] One way to think about analytic continuation is to remember that we can always pick a large, but finite, s-value along the positive s-axis. Since this point is regular, we can develop a Taylor series solution about that point. That Taylor series solution can be 'smooth pasted' with the one about the origin because their circles of convergence overlap. In fact, we use an ODE Solver to accomplish that – see the subsequent discussion.

$$G(s_2) \approx -\sqrt{\frac{\omega}{s_2}} + \frac{(\frac{\nu}{2} - \frac{1}{4} + c_1)}{s_2} + O(s_2^{-3/2}).$$

So, we use this initial condition to start an ODE solver at $s = s_2$ and develop the solution down to $s = s_1$, which gives us $G(s_1)$. But since $s = s_1$ lies inside our converge radius, we can apply there our previously developed power series solution $F(s; A) = F_1(s) + AF_2(s)$. The first derivative in that circle is of course given by $F'(s; A) = F_1'(s) + AF_2'(s)$. Then, we can solve the equation $G(s_1)F(s_1; A) = F'(s_1; A)$ for A. This solves our connection problem.

Finally, we need to do the Laplace inversion along some vertical contour $s = c + \mathrm{i}y$, where $c > 0$. Of course, the convenient choice at this point is $c = s_1$, and with that, we define $\tilde{G}(y) = G(c + \mathrm{i}y)$. Since $d/ds = -\mathrm{i}d/dy$, then the Riccati equation becomes

$$-\mathrm{i}\tilde{G}'(y) + [\tilde{G}(y)]^2 - \frac{\theta}{c + \mathrm{i}y}\tilde{G}(y) - \left[\frac{\omega}{c + \mathrm{i}y} + \frac{\lambda}{(c + \mathrm{i}y)(c + \mathrm{i}y + \eta)}\right] = 0. \qquad (22.23)$$

Using our ODE solver again, we can integrate (22.23) from $y = 0$ to arbitrary y, with $\tilde{G}(0) = G(s_1)$ from before. With the solution $\tilde{G}(y)$, then we obtain the analytic continuation of $F(s)$ in the vertical direction from the integration

$$F(c + \mathrm{i}y) = F(c) \exp\left\{\mathrm{i} \int_0^y \tilde{G}(t)\, dt\right\}. \qquad (22.24)$$

Finally, the stationary transition density is given by the Laplace inversion integral

$$\Psi(x) = \frac{1}{2\pi \mathrm{i}} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \mathrm{e}^{sx} F(s)\, ds = \frac{\mathrm{e}^{cx}}{\pi} \int_0^\infty \Re\left[\mathrm{e}^{\mathrm{i}yx} F(c + \mathrm{i}y)\right] dy. \qquad (22.25)$$

In the numerical implementation, this last integral is simply truncated at some sufficiently large value $y = y_{max}$ that achieves sufficient convergence.

Mathematica Notes. In Mathematica, the ODE solver is called `NDSolve` and it outputs an "interpolating function" object. Such functions are rapidly integrated by the `NIntegrate` function, which makes (22.24) a rapid evaluation. The connection constant problem, which requires the solution for A to $G(s_1)F(s_1; A) = F'(s_1; A)$ is handled by the `FindRoot` function. There are many implementation details.

Numerical example. With annual units, we take $\omega = 7/4$ and $\theta = 7/2$, so that the long run volatility, in the absence of jumps, is given by $\omega/\theta = 1/2$. For the jumps, we take $\lambda = 7$ and $\eta = 8$, which means that volatility jumps occur 7 times per year, on average, with a mean jump size of $\bar{\xi} = 1/\eta = 0.125$. These volatility jumps raise the long-run average volatility to

$$\bar{V}_\infty = \frac{1}{\theta}(\omega + \lambda\bar{\xi}) = \frac{3}{4}.$$

For the numerical implementation, after some experimentation, I settled on $s_1 = 1$ and $s_2 = 30$, with the Laplace inversion integral from $y = 0$ to $y = 200$. Various intermediate steps had to be performed at relatively high working precision. Our general procedure was to first ensure that we were reproducing the known results for $\lambda = 0$ with reasonable accuracy. With $\lambda = 0$, we produced a stationary density $\Psi(V)$ over the range $V = 0.1$ to $V = 5$. This density was small at both endpoints, had a normalization integral of 0.999 and a first moment $\bar{V} = 0.499$. In this case, it can be shown that the connection constant is given exactly by

$$A = (-1)\,\omega^\nu \,\frac{\Gamma[1-\nu]}{\Gamma[1+\nu]} = -\frac{49}{2025}\sqrt{7} \approx -0.0640206,$$

which was confirmed by the `FindRoot` routine to high accuracy.

Then, we turned on the jump parameters to the values indicated, and found a connection constant $A \approx -0.506504$. We generated a new density over the same range of V, finding a normalization integral of 0.999 and a first moment of $\bar{V} = 0.744$. It was difficult achieving accuracy in the tails of the distribution (for example, $V \geq 4$), although the density was quite small in this region. This difficulty is responsible for the relatively low accuracy in the computed value for $\bar{V}$, which should equal exactly 0.75. The density for the jumping case is plotted in Fig. 22.1 and numerical values for the density and its cumulative distribution are shown in Table 22.1. I estimate that the error in the cumulative probabilities is no worse than ± 0.002.

Conclusions. The general procedure used for the GARCH jump-diffusion, with exponential jumps, should handle a variety of other plausible jump distributions. All that was really necessary was the ability to construct a power series solution near $s = 0$, and a power series or asymptotic solution about $s = \infty$ for the Laplace transform of the stationary density. This will always be possible as long as $s = 0$ is, at worst, a regular singularity and $s = \infty$ is, at worst, an irregular singularity of finite rank of the associated ODE. With only positive volatility jumps, there will be no other singularities on the positive real s-axis, since $\hat{m}(s)$ is analytic for $\Re\, s > 0$.

Table 22.1. Stationary Density for a GARCH Jump-diffusion Process

Volatility V	Density $\Psi(V)$	Cumulative Probability
0.1	0.0018	0.0
0.25	0.736	0.038
0.5	1.370	0.352
1.0	0.470	0.798
1.5	0.137	0.932
2.0	0.046	0.973
3.0	0.0081	0.994
4.0	0.0021	0.998

Fig. 22.1. GARCH Jump-diffusion: Stationary Density