Introduction to the XGBM Model

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Abstract

I give an introduction to "Exact Solutions for a GBM-type Stochastic Volatility Model having a Stationary Distribution", my article which appears elsewhere in this issue.

1 A model with some nice properties

XGBM is short for "Extended Geometric Brownian Motion". The model is developed in "Exact Solutions for a GBM-type Stochastic Volatility Model having a Stationary Distribution", which appears in this issue. Here I add motivation, focusing on time series fits. The full article is more focused on option valuation.[1]

It's a stochastic volatility model. If you're new to those, first consider the basic model for stock prices:[2]

$$dS_t = \mu S_t \, dt + \sigma S_t \, dB_t. \tag{1}$$

The drift μ and volatility σ are constants; B_t is a Brownian motion process.

Stochastic volatility models generalize (1) by promoting the volatility parameter to a stochastic process: $\sigma \to \sigma_t$. Typically, they also generalize the drift. For time-homogeneous models, the result is

$$dS_t = \mu(\sigma_t) S_t \, dt + \sigma_t S_t \, dB_t. \tag{2}$$

In addition, one needs an SDE for σ_t, or equivalently for the instantaneous variance rate $v_t = \sigma_t^2$. I take:

$$\mathbb{P}\text{-XGBM}: \quad \begin{cases} dS_t = (\alpha + \beta\, \sigma_t^2) S_t \, dt + \sigma_t S_t \, dB_t, \\ d\sigma_t = \sigma_t(\omega - \theta\, \sigma_t)\, dt + \xi\, \sigma_t \, dW_t, \\ dB_t \, dW_t = \rho \, dt. \end{cases} \tag{3}$$

where now (B_t, W_t) are two *correlated* Brownian motions.

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[1] This is the familiar $\mathbb{P}$-model vs. $\mathbb{Q}$-model dichotomy in finance.

[2] The basic model SDE (stochastic differential equation) is the Black-Scholes-Merton model in the option context. It's due to M.F.M. Osborne in 1959 in the time series context.

Financial evolution models come in two versions: the $\mathbb{P}$-version and the $\mathbb{Q}$-version. The $\mathbb{P}$-version is the "real-world" model. Parameters are found by time series fits, in this case $(\alpha, \beta, \omega, \theta, \xi, \rho)$, using exact or approximate Maximum Likelihood Estimation (MLE). That explains the $\mathbb{P}$-XGBM in (3). For option and other derivative securities valuation, one uses the $\mathbb{Q}$-version, the "risk-neutral model". It's not given here, but found in the full article.

If X_t is an arbitrary process and $dX_t \sim a\, X_t\, dW_t$, (ignoring the drift), let's call it "GBM-type". The "GBM-type" in my article's title refers to the volatility equation (second SDE) in (3). As one sees, $d\sigma_t \sim \xi \sigma_t\, dW_t$. In brief, here are some nice properties of the model:

- It has GBM-type volatility.
- It has a stationary density, $\psi(\sigma_t)$, for the volatility.
- The $\mathbb{P}$-version has an exact solution for the bivariate transition density.
- The $\mathbb{Q}$-version has an exact solution for option values.

What's attractive about GBM-type volatility? First, you find some improvement in time series fit over the competing alternative of the '93 Heston model. The Heston model is the "base case" stochastic volatility model: arguably, the most important model in Quantitative Finance after the Black-Scholes-Merton model. Indeed, in Sec. 2, I show some limited fits to the stationary volatility density. If you want to fit all the parameters, see Sec. 4.

Dropping the drifts in (3) leaves the lognormal SABR model, well-known to readers of this magazine (Hagan, Kumar, Lesniewski, & Woodward, 2002). *All* GBM-type volatility models (including SABR) behave like "Brownian motion on the hyperbolic plane" at small times. It seems obscure, but see Sec. 4 for why it's a helpful property.

2 Fitting some stationary volatility densities

Let's fit $\psi(\sigma_t)$ for XGBM and competing models to time series. Most popular by far is the Heston '93 model. Like XGBM model, it has a stationary volatility density, and exact solutions for both the bivariate transition density ($\mathbb{P}$-version) and option values ($\mathbb{Q}$-version). It does *not* have GBM-type volatility, but instead the v_t evolution has the well-known "square-root model" form:

$$\text{Heston '93:} \quad dv_t = (\omega - \theta\, v_t)\, dt + \xi\, \sqrt{v_t}\, dW_t. \tag{4}$$

You may object that we've switched from σ_t to $v_t = \sigma_t^2$, but if an SDE is GBM-type under one, it will be GBM-type under the other. That follows from Itô's transformation rule. Thus, the fact that (4) is *not* GBM-type under v_t means that it certainly isn't under σ_t either.

Another favorite of mine is the GARCH-diffusion model:

$$\text{GARCH-diffusion:} \quad dv_t = (\omega - \theta\, v_t)\, dt + \xi\, v_t\, dW_t. \tag{5}$$

This one *does* have GBM-type volatility, and a stationary density. Unfortunately, it lacks an exact solution, although it's certainly amenable to numerics. Finally, consider the "3/2-model": $dv_t = v_t(\omega - \theta\, v_t)\, dt + \xi\, v_t^{3/2}\, dW_t$. That one has a stationary density and exact solutions, but it's not GBM-type.

Finding the various $\psi(\sigma_t)$ is routine. Note that for the models written with v_t, you might want to first develop $\psi_V(v_t)$ and then make an easy change-of-variable.

These are all 3-parameter probability densities. I report Maximum Likelihood Estimation (MLE) fits to a proxy series for $\{\sigma_t\}$, done previously for my blog.[3] The proxy is the (annualized) daily volatility for the S&P500 Index, taken from the Oxford-Man Institutes' "realized library"[4]. They maintain a number of estimators for σ_t – I use their basic (rv5), the daily volatility estimated from 5-minute log-return observations. All the data available at the time was Jan 3, 2000 - Sept 28, 2018: 4705 volatility observations. Note rv5 does not include volatility from the overnight move in the index. Including, say, a squared-return for that would be a good idea for full model MLE. But we can ignore the overnight for simple $\psi(\sigma_t)$ compares – our only goal here.

Fig. 1 shows the volatility time series. There's a maximum around 140%, an annualized volatility at the height of the 2007-2008 Financial Crisis.

Fig. 2 shows the stationary density fits for the 4 models. Each figure shows the same realized volatility histogram, overlaid with the MLE fit to the labeled model. As it turns out, while the GARCH diffusion and the 3/2-model are generally different, they have same functional form for $\psi(\sigma_t)$. That's why the last figure is labeled 'GARCH-3/2 model".

For Fig. 2, I am using the annualized volatility in decimal (not %), so the time series maximum is a histogram entry at $\sigma \approx 1.4$. That small bump is not really visible, but is accounted for by the fits and suggested by the axis limit.

As you can see, the visual fit is better for the XGBM model as compared to the Heston model, with corresponding log-likelihoods: 5356 (Heston) vs. 5920 (XGBM). This suggests, but does not prove, that the XGBM model is going to also do better than the Heston model in full MLE with joint transition densities.

As one sees, the $\psi(\sigma_t)$ fit is even better (LL = 6055) for the the GARCH-3/2 model. Remember GARCH is also GBM-type, but the 3/2 model is not. These last two models would be distinguished by a joint density MLE compare. There are further cites in the main article to model compares.

Regardless of how that joint density compare might turn out, let's recap the modest point here: XGBM bests Heston '93 model in a fit of $\psi(\sigma_t)$ to our proxy.

3 Background on the model solution method

Model solutions are found by a spectral expansion method. The model is closely related to a one-factor interest rate model of R. C. Merton's:

[3]See my Oct. 8, 2018 blog entry at `www.financepress.com`

[4]`https://realized.oxford-man.ox.ac.uk/`

$$dr_t = r_t(\omega - \theta\, r_t)\, dt + \xi\, r_t\, dW_t. \tag{6}$$

Indeed r_t and XGBM's σ_t have the same SDE. The solution target with interest rates is the discount bond price $B(r,t)$, which solves the partial differential equation problem:

$$\text{Merton's model}: \quad \begin{cases} B_t = \frac{1}{2}\xi^2 r^2 B_{rr} + (\omega r - \theta r^2) B_r - rB, \\ B(r,0) = 1, \end{cases} \tag{7}$$

using subscripts for partial derivatives. I gave the full (spectral method) solution to this problem in (Lewis, 1998), so that's key background.

For spectral expansion novices, I gave "Tips and Tricks", mostly applying to 1D diffusions, in Ch. 2, pg 54 of (Lewis, 2016). Some tips relevant here:

- Suppose you want to characterize the spectrum for a 1D diffusion with generator $\mathcal{A}f = \frac{1}{2}a^2(x)f'' + b(x)f' - c(x)f$ *without actually computing it.* A cookbook:

  Step 1. Create the fundamental scale, speed, and killing densities; compute the Feller boundary classifications. Is each boundary either regular, entrance, or exit? If 'yes', the spectrum is purely discrete, by (Elliott, 1955) and (McKean, 1956), and you are done. If not, move on to Step 2.

  Step 2. Transform $(\mathcal{A} + \lambda)f = 0$ to Liouville normal form/Schrödinger equation form $U'' + [\lambda - q(x)]U = 0$. Based upon that coordinate transformation and the resulting Schrödinger 'potential' $q(x)$, apply 'Oscillation Theorems' to characterize the spectrum (Linetsky, 2004).

- How can you know if a given problem has a spectral expansion in terms of special functions? First, experience. Second, transformation to standard forms. Finally, a good trick is to simply give (Mathematica's) `DSolve` the Green function ODE $(\mathcal{A} - s)u = 0$. If you get an answer in terms of built-in special functions, you're good to go. In my experience, this is almost always a reliable guide.

- Get used to continuous and mixed spectrum cases. Study examples in (Lewis, 2016), (Titchmarsh, 1962), or anywhere else you can find them.

- Learn to love the confluent hypergeometric functions $M(a,b,z)$ and $U(a,b,z)$. A surprising number of solvable spectral problems reduce to solving Kummer's differential equation.

- Spectral expansions can be difficult to use at small-T. You may need a separate small-T asymptotic solution.

4 Remarks on full model parameter fits

The $\psi(\sigma_t)$ fits do not resolve all the parameters of (3). In particular, we're missing β, arguably the most important because it yields the equity risk premium.

Better would be a comparative analysis via complete MLE. By complete MLE, I mean maximizing $LL(\Theta)$, where LL is the log-likelihood of a time series and Θ is the full parameter set. Let's relabel the time series $\{x_t, y_t\}$, where $x_t = \log S_t$ and $y_t = \sigma_t$.

Again, a good proxy for $\{y_t\} = \{\sigma_t\}$ is likely the Oxford-Man Realized Library estimates, supplemented with an overnight volatility correction. Full MLE requires the joint bivariate transition densities. Let's write boldface $\mathbf{x_t} = (x_t, y_t)$; assume daily observations. Then MLE is

$$\max_{\Theta} LL(\Theta) = \max_{\Theta} \sum_n \log p_t(\mathbf{x}(t_{n+1})|\mathbf{x}(t_n)), \tag{8}$$

where the dependence of the p_t on Θ has been suppressed.

The joint transition density for the XGBM model is found at eqn (11) in my article. For the Heston and 3/2-model, risk-neutral transition densities are found in Ch. 7 of (Lewis, 2016). Those cited formulas are readily adapted to the $\mathbb{P}$-versions where the stock price drift takes the same form as in (3).

However, no closed-form p_t is available for the GARCH-diffusion – instead, use small-time expansions. Indeed, even when closed-form densities are available, you may want to adopt small-time expansions. That's because, as we noted above, spectral expansions can be difficult at small times. Use available closed-forms to check on the accuracy of any putative small-time approximation.

The small-time expansion for the joint transition density of general time-homogeneous diffusions is also known as the "heat-kernel expansion". For two-dimensional diffusions, it takes the form:

$$p_t(\mathbf{x}'|\mathbf{x}) = \frac{1}{2\pi t}\, \mathrm{e}^{-d^2(\mathbf{x},\mathbf{x}')/2t} \times (a_0 + a_1 t + a_2 t^2 + \cdots). \tag{9}$$

The various coefficients a_i depend upon the two coordinates $(\mathbf{x}, \mathbf{x}')$ but not t. Most importantly, in a classic result due to the probabilist S.R.S. Varadhan, $d(\mathbf{x}, \mathbf{x}')$ has the interpretation as a geodesic distance. Specifically, its a minimal distance in a geometry in which the metric is given by the inverse of the variance-covariance matrix of the system. For the two GBM-type models we have considered (XGBM and GARCH-diffusion), and with some standardization, this metric can be shown to be

$$(ds)^2 = \frac{(dx)^2 + (dy)^2}{y^2}. \tag{10}$$

That is the famous metric of the Poincaré half-plane model of hyperbolic geometry. The associated distance function is explicit, and relatively simple for the class of models we are considering:

$$d(\mathbf{x}, \mathbf{x}') = \cosh^{-1}\left(1 + \frac{(x - x')^2 + (y - y')^2}{2yy'}\right). \tag{11}$$

As a practical matter, beyond the leading terms, the HKE in (9) rapidly becomes very difficult as written. Instead, you'll want to employ a *double* power series expansion in which the various a_i are themselves expanded in Taylor series using powers of $(x - x')$ and $(y - y')$. This double expansion is capable of being automated in a good symbolic algebra system, such as Mathematica. For some details of this last approach, see (Ait-Sahalia & Kimmel, 2007).

Figure 1: **Oxford-Man daily volatility (rv5) for the S&P500 index**

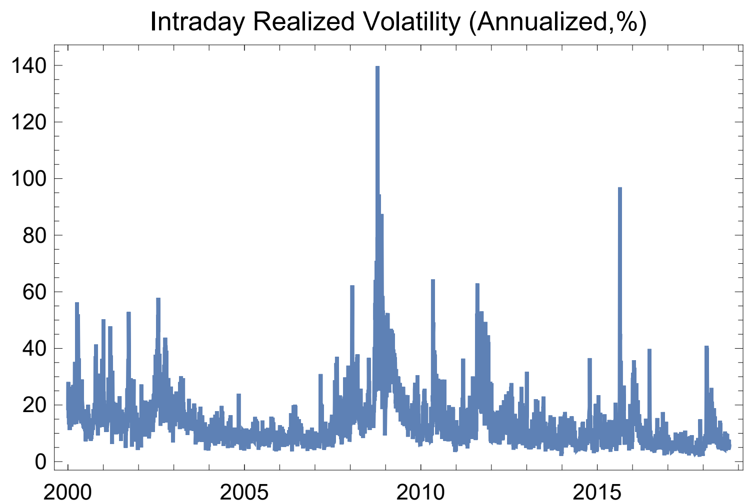
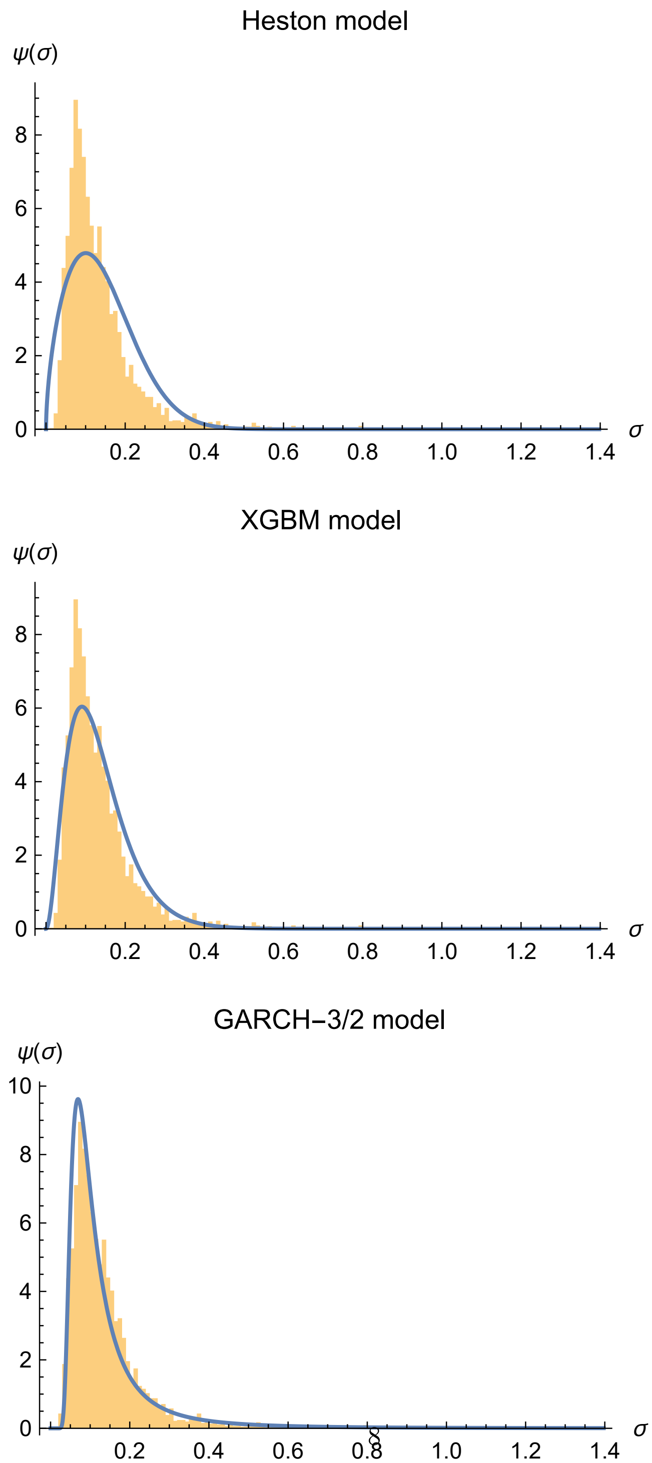


Figure 2: **Stationary Volatility Distribution Fits for 4 Models**



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