American - style options in the Black - Scholes model
Handle discrete dividends, using the 'Back to Basics' piecewise GBM theory and NDSolve.
Handles any Div policy function

Example for Vol II, Ch .9
Created under Ver 9.0
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Documentation.
For code explainations, further documentation, see
"Option Valuation under Stochastic Volatility II: with Mathematica Code",
Alan L. Lewis, (2016) Finance Press, Newport Beach, California
Chapter 9 : "Back to Basics: An Update on the Discrete Dividend Problem"

Notes :
Vplus (S) = P (S, td -)
Vminus (S) = P (S, td +)

Important : Early exercise at intra - dividend times, is checked every dt = 1/nexoppsperyear,
where the latter is the 'number of exercise opportunities per year'. (Using 250 mostly here).
When choosing ex - dividend times, you MUST choose times that are integer multiples of dt, or they will
be missed by the solver

Clear[MIU, MMU];
MIU := N[MemoryInUse[] / 10^9];
MMU := N[MaxMemoryUsed[] / 10^9];

(* Every dt=1/nexoppsperyear,
check for soln falling below initi soln, and adjust *)
(* The PDE time t = T - t, where t = calendar time *)
(* The first dividend is at td1 in pde time *)
(* The interval between dividends is tdint in pde time *)
(* So there is a discrete dividend Div,
whenever the calendar time t=T-td1, T-td1-tdint, T-td1-2 tdint, etc *)
(* So, there may be 0 or multiple dividends per year *)
(* The solver is forced to take at least 10 tsteps between events *)
(* Uses global DivPolicy[Div] *)
Clear[solverDivPolicy];
solverDivPolicy[S0_, K_, td1_, tdint_, Div_,
  nexoppsperyear_, r_, initsoln_, T_, npts_, rpts_, pflag_] :=
  Module[{Smap, xmap, xgrid, Soln, h, xmin, xmax, dt, ecnt = 0,}
divflag, divecnt = 0, nextTd = tdl, hsoln, DP),

xgrid = N[Table[i / npts, {i, 0, npts - 1}]]; (* Keep away from 1 *)
xmin = xgrid[[1]];
xmax = xgrid[[npts]];
dt = N[1 / noppssperyear];

Smap[x_] := S0 x / (1.0 - x);
xmap[s_] := s / (s + S0);

DP[S_] = DivPolicy[Div, S];
Print["Solver: using div policy D(S)=",
DP[S], " tdl=", tdl, " tdint=", tdint, " T=", T];

Clear[So, h];
Soln =

h /. NDSolve[{h[x, τ] = a1[x] ∂x,xh[x, τ] + b1[x] ∂x,xh[x, τ] - r h[x, τ],
    h[x, 0] = initsohn[x],
    h[xmin, τ] = initsohn[xmin], h[xmax, τ] = initsohn[xmax],
    WhenEvent[Mod[τ, dt] == 0, ecnt++; divflag = 0];
    If[(divecnt == 0 && Chop[τ - tdl] == 0) || (divecnt > 0 && Chop[nextTd - τ] == 0),
        Print["Div detected at τ=", τ];
        divflag = 1; divecnt++; nextTd += tdint, Null];
    hsoln[x_] := If[divflag == 1, h[xmap[Smap[x] - DivPolicy[Div, Smap[x]]], τ],
        Max[h[x, τ], initsohn[x]]];
    h[x, τ] -> Outer[hsoln[#1] & , xgrid]], {h}, {x, xmin, xmax},
    {τ, 0, T}, (* MUST USE {τ,0,T} *)
StepMonitor :> (ct++; If[pflag == 1 && Mod[ct, rpts] == 0,
            Print["solverEvent: ct=", ct, " τ=", τ, " MIU(GB)=" , MIU, " Null]),
        MaxSteps -> 1000 000, MaxStepFraction -> Min[dt / (10 T), 1 / npts],
        Method -> {"MethodOfLines",
            "SpatialDiscretization" -> {"TensorProductGrid", "Coordinates" -> xgrid}]][[1]];

Print["solverDivPolicy: done; T=" , T, " found ", ecnt, 
" early exercise events including ", divecnt, " discrete dividend events"];
Return[Soln]]

(* American-style Put Values under BS Model *)
(* Allows NDSolve on unit square coords with WhenEvent *)
(* The first dividend prior to expiration is at T-Tdl, and the interval between dividends is Tdint *)
(* noppssperyear = number of Bermudan exercise opps per year; suggest 250 *)
(* You MUST have ex-div dates lying exactly on a Bermudan exercise time *)
(* So, if noppssperyear = 1/250 = 0.004, 
then ex-div times must be exactly divisible by 0.004 *)
(* Requires Td1 < T; but if you want no dividends, set Div=0 *)
Clear[PutDivPolicy];
PutDivPolicy[S0_, K_, T_, r_, sig0_, pflag_, rpts_, solverpflag_,
  npts_, noppsperyear_, Td1_, Tdint_, Div_, tinset_, size_] :=
Module[{C0, xmin, xmax, Smap, V = sig0^2, initso1n, soln, ans, eps, divx, dt,
  xgrid, xmap, x0, x1, t1, ans0, MyTimeValue, S1, teval, Scrit,
  p, plotinset, point1, point2, label, Tdlast},
  dt = 1/(10 noppsperyear); (* Max dt that the solver will use *)
  If[T = 0, Return[Max[K - S0, 0]], Null];

  (* Unit square coords *)
  C0 = N[S0];
  xmap[s_] := s / (s + C0);

  Clear[Smap];
  Smap[x_] := C0 x / (1.0 - x);

  eps = 1.0 / npts;
  xmin = eps;
  xmax = 1.0 - eps;

  (* PDE coefs for x-derivatives: GLOBALS *)
  Clear[a1, b1];
  a1[x_] = Simplify[0.5 V x^2 (1.0 - x)^2];
  b1[x_] = Simplify[(r - V x) x (1.0 - x)];

  (* Hotspot *)
  x0 = 0.5;

  (* Insert the dividend policy function here, with chart label *)
  Clear[DivPolicy]; (* Global *)
  DivPolicy[divx_, Sx_] = Min[divx, Sx]; (* HHL Liquidator *)
  label = "D(S)=" <> "Min[" <> ToString[Div] <> ",S];"

  (* HHL Survivor *)
  label = "D(S)=" <> ToString[Div] <> ",1(S)";

  (* initial soln *)
  initso1n[x1_] = Max[K - Smap[x1], 0];

  Off[NDSolve:::erri];
  Clear[ct]; ct = 0;
  soln = solverDivPolicy[S0, K, Td1, Tdint,
    Div, noppsperyear, r, initso1n, T, npts, rpts, solverpflag];

  ans0 = soln[x0, T];
  Print["PutDivPolicy:ct"," x0=" x0, " npts=" npts,
    " noppsperyear=" noppsperyear, " Price=" ans0, " (MMU=" MMU, " GB)" ];
If[pflag == 0, Return[ans0], Null];

(* Plot soln starting from t=0 to tinset *)
plotinset = 
Plot[soln[x0, T - t1], {t1, 0, Min[tinset, T]}, AxesLabel -> {"t", "P(S=100,t)"},
    ImageSize -> 150];

MyTimeValue[x1_, t1_] := soln[x1, t1] - Max[K - Smap[x1], 0];
xxgrid = N[Table[i/npts, {i, 0, npts - 1}]]; (* Keep away from 1 *)

Clear[GenericTV, GenericScrit, critdata];
GenericTV[x1_, t1_] := MyTimeValue[x1, T - t1];
GenericScrit[t1_] := If[t1 == T, K, Smap[GenericXcritSearch[xxgrid, t1]]];
Print["At t=0 found Scrit=" , GenericScrit[0]]; (* Now at tD-1, inst prior to going ex-dividend for the last time, counting backwards from expiration *)
Tdlast = Td1; 
While[Tdlast <= T, Tdlast += Tdint];
Tdlast -= Tdint;

(* Now at tD-, inst prior to going ex-dividend for the last time, counting backwards from expiration *)
teval = Tdlast + dt;
Scrit = GenericScrit[T-teval];
Print["At t=" , T-teval, " with dx=" , eps, " found Scrit=" , Scrit];

Print["Creating Vpluschart at t=" , T-teval];
Clear[Vpluschart];
Vpluschart = 
Plot[soln[xmap[S1], teval], {S1, Smap[xmin], 1.2 K}, ImageSize -> size,
    FrameLabel -> {"V^+(S)", Null}, {"S", label}], Frame -> True,
    Epilog -> Inset[plotinset, (87, 75)]]; (* Now at tD+, just after going ex-dividend *)
teval = Tdlast - dt;
Print["Creating Vminuschart at t=" , T-teval];
Clear[Vminuschart];
Vminuschart = 
Plot[soln[xmap[S1], teval], {S1, Smap[xmin], 1.2 K}, ImageSize -> size,
    FrameLabel -> {"V^-(S)", Null}, {"S", label}], Frame -> True];

Scrit = GenericScrit[T-teval];
Print["At t=" , T-teval, " with dx=" , eps, " Scrit=" , Scrit];

point1 = Point[Scrit, K-Scrit];
point2 = Point[Scrit + Div, K-Scrit];
Clear[CombinedChart];
CombinedChart = Show[Vpluschart, Vminuschart,
  Graphics[point1], Graphics[point2], ImageSize -> size];

critdata = Table[{t1, GenericScrit[t1]}, {t1, 0, T, T/1000}];
p = ListPlot[critdata, PlotRange -> {0, K},
  Filling -> Axis, Joined -> True, ImageSize -> size,
  FrameLabel -> {"S_crit", Null}, {"t", label}}, Frame -> True];

Return[GraphicsColumn[{p, CombinedChart}]]; }

(* Return interpolated value of x at first zero crossing pt *)
Clear[GenericXcritSearch];
GenericXcritSearch[xgrid_, t1_] := Module[{tvfunc, len, xmin, f1, f2, m, x0, x1, x2, xcrit, dx},
  len = Length[xgrid];
  dx = 1.0/len;
  xmin = xgrid[[2]]; 
  xcrit = xgrid[[1]]; 
  tvfunc[xx_] := GenericTV[xx, t1];
  f1 = tvfunc[xmin];
  If[f1 > 0, Return[xcrit], Null];

  x1 = xmin;
  For[x2 = xmin + dx, x2 < 0.5, x2 += dx, 
    f2 = Chop[tvfunc[x2]]; 
    m = (f2 - f1) / dx;
    If[f2 > 0, xcrit = -f1 / m + x1; Break[], Null];
    x1 = x2;
  ];
  Return[xcrit]

(* HHL Liquidator policy with Div = 4, (td1,taint) = (0.4,1) *)
PutDivPolicy[100, 100, 5, 0.08, 0.40, 1, 20000, 1, 250, 250, 0.4, 1, 4, 5, 400]
Solver: using div policy $D(S)=\text{Min}[4, S]$ $\text{td}_1=0.4$ $\text{td}_2=1$ $T=5$

Div detected at $t=0.4$
Div detected at $t=1.4$
Div detected at $t=2.4$
Div detected at $t=3.4$
Div detected at $t=4.4$

solverDivPolicy: done; $T=5$ found 1250 early exercise events including 5 discrete dividend events

PutDivPolicy: $c_t=13795$ $x_0=0.5$ $\text{npts}=250$ $\text{noppsperyear}=250$ $\text{Price}=24.4582$ ($\text{MMU}=0.194909$ GB)
At $t=0$ found $S_{\text{crit}}=43.2251$
At $t=0.5996$ with $dx=0.004$ found $S_{\text{crit}}=0$.
Creating $V_{\text{plus}}$ chart at $t=0.5996$
Creating $V_{\text{minus}}$ chart at $t=0.6004$
At $t=0.6004$ with $dx=0.004$ found $S_{\text{crit}}=49.6711$

![Graph of $D(S)=\text{Min}[4, S]$](image1)

![Graph of $P(S)$](image2)
(* HHL Survivor policy with Div = 4, (td1,tdint) = (0.4,1) *)
PutDivPolicy[100, 100, 5, 0.08, 0.40, 1, 20000, 1, 250, 250, 0.4, 1, 4, 5, 400]
Solver: using div policy D(S)=If[S < 4, 0, 4] td1=0.4 tdint=1 T=5
  Div detected at t=0.4
  Div detected at t=1.4
  Div detected at t=2.4
  Div detected at t=3.4
  Div detected at t=4.4
 solverDivPolicy: done; T=5 found 1250
  early exercise events including 5 discrete dividend events
PutDivPolicy: ct=13795 x0=0.5 npts=250 noppsperyear=250 Price=24.4582 (MMU=0.316435 GB)
At t=0 found Scrit=43.2251
At t=0.5996 with dx=0.004 found Scrit=3.73689
Creating Vpluschart at t=0.5996
Creating Vminuschart at t=0.6004
At t=0.6004 with dx=0.004 Scrit=49.6711
(* Book example: HHL Liquidator with Div = 4, (td1,tdint) = (0.4,1) *)
PutDivPolicy[100, 100, 5, 0.08, 0.40, 0, 20000, 1, 250, 250, 0.4, 1, 4, 5, 400]
Solver: using div policy D(S)=Min[4, S] td1=0.4 tdint=1 T=5
Div detected at \( t=0.4 \)
Div detected at \( t=1.4 \)
Div detected at \( t=2.4 \)
Div detected at \( t=3.4 \)
Div detected at \( t=4.4 \)
solverDivPolicy: done; T=5 found 1250
early exercise events including 5 discrete dividend events
PutDivPolicy:ct=13795 x0=0.5 npts=250 noppsperyear=250 Price=24.4582 (MMU=0.711901 GB)
24.4582

PutDivPolicy[100, 100, 5, 0.08, 0.40, 0, 20000, 1, 250, 500, 0.4, 1, 4, 5, 400]
Solver: using div policy $D(S) = \text{Min}[4, S]$ $td1=0.4$ $tdint=1$ $T=5$

Div detected at $t=0.4$
Div detected at $t=1.4$
Div detected at $t=2.4$
Div detected at $t=3.4$

solverEvent: $ct=20000$ $t=3.6202$ $\text{MIU(GB)}=0.11228$

Div detected at $t=4.4$

solverDivPolicy: done; $T=5$ found 2500
  early exercise events including 5 discrete dividend events

PutDivPolicy: $ct=27682$ $x0=0.5$ $\text{npts}=250$ $\text{noppsperyear}=500$ $\text{Price}=24.4631$ ($\text{MMU}=0.367804$ $\text{GB})$

24.4631